

# PINCHUK MAPS, FUNCTION FIELDS, AND REAL JACOBIAN CONJECTURES

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ABSTRACT. All counterexamples of Pinchuk type to the strong real Jacobian conjecture (SRJC) are shown to have function field extensions of degree six with no nontrivial automorphisms. Real Jacobian conjectures are considered for rational, as well as polynomial, maps, with nonrational inverses allowed in both cases. The birational and Galois cases are highlighted. Modifications, in particular to the SRJC, that exclude the Pinchuk counterexamples, are suggested.

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## 1. INTRODUCTION

The Jacobian Conjecture (JC) asserts that a polynomial map  $F : k^n \rightarrow k^n$ , where  $k$  is a field of characteristic zero, has a polynomial inverse if its Jacobian determinant,  $j(F)$ , is a nonzero element of  $k$ . The JC is still not settled for any  $n > 1$  and any specific field  $k$  of characteristic zero. It is known, however, that if it is true for  $k = \mathbb{C}$  and all  $n > 0$ , then it is true in every case. As  $j(F)$  is the determinant of the Jacobian matrix of partial derivatives of  $F$ , it is polynomial,

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and so for  $k = \mathbb{C}$  it is a nonzero constant if, and only if, it vanishes nowhere on  $\mathbb{C}^n$ . That suggested the Strong Real Jacobian Conjecture (SRJC), which asserts that a polynomial map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , has a real analytic inverse if  $j(F)$  vanishes nowhere on  $\mathbb{R}^n$ . However, Sergey Pinchuk exhibited a family of counterexamples for  $n = 2$ , now usually called Pinchuk maps.

Say  $F = (f_1, \dots, f_n)$ , with each component a polynomial in  $x_1, \dots, x_n$ . If  $j(F)$  is not identically zero, the components of  $F$  are algebraically independent over  $k$ , so  $k[F] = k[f_1, \dots, f_n] \subseteq k[X] = k[x_1, \dots, x_n]$  is an inclusion of polynomial algebras. The extension of function fields  $k(F) \subseteq k(X)$  is algebraic, since both fields have transcendence degree  $n$  over  $k$ , and is finitely generated, hence of finite degree. In the JC context with  $j(F)$  a nonzero constant,  $F$  has a polynomial inverse, and hence  $k[F] = k[X]$ , if the extension  $k(F) \subseteq k(X)$  is Galois, in particular in the birational case  $k(F) = k(X)$ . For  $k = \mathbb{R}$  and the SRJC context, that yields polynomial invertibility in both the birational and Galois cases if  $j(F)$  is a nonzero constant; but apparently there are no published invertibility results in either case if  $j(F)$  just vanishes nowhere on  $\mathbb{R}^n$ .

In section 2 the extension of function fields is investigated for a previously well studied Pinchuk map. A primitive element is found, its minimal polynomial is calculated, and the degree (6) and automorphism group (trivial) of the extension are determined. That generalizes to any Pinchuk map  $F$  defined over any subfield  $k$  of  $\mathbb{R}$ . Although  $F$  is generically two to one as a polynomial map of  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , the degree of the associated extension of function fields  $k(F) \subset k(X)$  is 6 and  $k(X)$  admits no nontrivial automorphism that fixes all the elements of  $k(F)$  (Theorem 2.1). In particular, the extension is not Galois.

Section 3 treats the more general case of real rational everywhere defined maps  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with nowhere vanishing Jacobian determinant and their associated function field extensions. If the extension is birational, then  $F$  has an inverse of the same character as  $F$  (Theorem 3.2). If it is Galois, then it is birational if  $F$  has a real analytic inverse (Theorem 3.8). Two necessary conditions for invertibility are found to apply to the extension: trivial automorphism group and odd degree. The degree parity restriction produces modified conjectures, in particular a new variant of the SRJC, with hypotheses that exclude the Pinchuk counterexamples. Theorems 3.9 and 3.10 prove some reductions to special cases of the sort familiar in the ordinary JC context.

Section 4 is an appendix. It contains additional detailed information on the specific Pinchuk map of section 2 that is not needed for the proofs there, but can be used to verify assertions about the map.

The exposition is often at an elementary level, which the author hopes will be excused by sophisticated readers.

## 2. PINCHUK MAPS

Pinchuk maps are certain polynomial maps  $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that have an everywhere positive Jacobian determinant  $j(P, Q)$ , and are not injective [1]. The polynomial  $P(x, y)$  is constructed by defining  $t = xy - 1$ ,  $h = t(xt + 1)$ ,  $f = (xt + 1)^2(t^2 + y)$ ,  $P = f + h$ . The polynomial  $Q$  varies for different Pinchuk maps, but always has the form  $Q = q - u(f, h)$ , where  $q = -t^2 - 6th(h + 1)$  and  $u$  is an auxiliary polynomial in  $f$  and  $h$ , chosen so that  $j(P, Q) = t^2 + (t + f(13 + 15h))^2 + f^2$ .

### 2.1. A specific Pinchuk map.

The specific Pinchuk map used in this paper to investigate the associated extension of function fields is one introduced by Arno van den Essen via an email to colleagues in June 1994. It is defined [2] by choosing

$$(1) \quad u = 170fh + 91h^2 + 195fh^2 + 69h^3 + 75fh^3 + \frac{75}{4}h^4.$$

The total degree in  $x$  and  $y$  of  $P$  is 10 and that of  $Q$  is 25. The image, multiplicity and asymptotic behavior of  $F$  were studied in [3, 4, 5, 6]. Its asymptotic variety,  $A(F)$ , is the set of points in the image plane that are finite limits of the value of  $F$  along curves that tend to infinity in the  $(x, y)$ -plane [7, 8]. It may alternatively be defined as the set of points in the image plane that have no neighborhood with a compact inverse image under  $F$  [9, 10, 11]. It is a topologically closed curve in the image  $(P, Q)$ -plane and is the image of a real line under a bijective polynomial parametrization; Its Zariski closure has one additional point not on the curve, so it is a semi-algebraic variety, but not an actual real algebraic variety. It is depicted below using differently scaled  $P$  and  $Q$  axes. It intersects the vertical axis at  $(0, 0)$  and  $(0, 208)$ . Its leftmost point is  $(-1, -163/4)$ , and that is the only singular point of the curve.

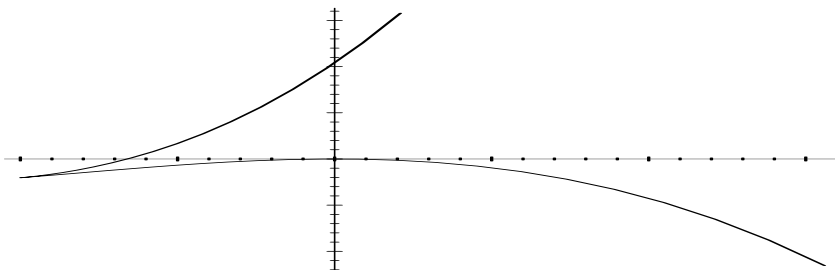


FIGURE 1. The asymptotic variety of the Pinchuk map  $F$ .

The points  $(-1, -163/4)$  and  $(0, 0)$  of  $A(F)$  have no inverse image under  $F$ , all other points of  $A(F)$  have one inverse image, and all points of the image plane not on  $A(F)$  have two.

The inverse image of  $A(F)$  under  $F$  is the disjoint union of three smooth curves, each of which is a topological line that extends to infinity at both of its ends. The curves partition their complement in the  $(x, y)$ -plane into four simply connected unbounded open sets. Those regions are mapped homeomorphically to their images, two each to the regions on either side of  $A(F)$ .

### 2.2. Minimal polynomial calculation.

This paragraph is a summary of some key facts from previously cited work on  $F$ . A general level set  $P = c$  in the  $(x, y)$ -plane has a rational parametrization.

Specifically, for any real  $c$  that is not  $-1$  or  $0$ , the equations

$$x(h) = \frac{(c-h)(h+1)}{(c-2h-h^2)^2}$$

$$y(h) = \frac{(c-2h-h^2)^2(c-h-h^2)}{(c-h)^2},$$

define a rational map pointwise on a real line with coordinate  $h$ , except where a pole occurs. The use of  $h$  as a parameter and the equality  $P = c$  are consistent: on substitution into the defining equations  $t = xy - 1$ ,  $h = t(xt + 1)$ ,  $f = (xt + 1)^2(t^2 + y)$ ,  $P = f + h$ , the expression  $h(x(h), y(h))$  simplifies to  $h$ , and  $P(x(h), y(h))$  to  $c$ . There is always a pole at  $h = c$  and  $Q(x(h), y(h))$  tends to  $-\infty$  as the pole is approached from either side. Also,  $Q(x(h), y(h))$  tends to  $+\infty$  as  $h$  tends to  $+\infty$  and as  $h$  tends to  $-\infty$ . If  $c > -1$ , there are two additional poles at  $h = -1 \pm \sqrt{1+c}$  and  $Q(x(h), y(h))$  tends to a finite asymptotic value at each of these poles as the pole is approached from either side. In that case the asymptotic values are distinct and are the values of  $Q$  at the two points of intersection of the vertical line  $P = c$  and  $A(F)$  in the  $(P, Q)$ -plane. The level sets  $P = c$  are disjoint unions of their connected components, which are curves that are smooth (because of the Jacobian condition) and tend to  $\infty$  in the  $(x, y)$ -plane at both ends. The number of curves is two if  $c < -1$ , and four if  $-1 < c \neq 0$ . Even the two exceptional values fit this pattern, although they require different rational parametrizations, with  $P = -1$  consisting of four curves and  $P = 0$  of five.

As a concrete illustration, consider the case  $P = 3$ . Detailed justifications are omitted. The points of intersection of the vertical line  $P = 3$  and  $A(F)$  are  $a = (3, 14965/4)$  and  $b = (3, -4235/4)$ . The poles are at  $h = -3$ ,  $h = 1$ , and  $h = 3$ . As  $h$  varies from  $-\infty$  to  $3$  the image point  $F(x(h), y(h))$  moves down the vertical line  $P = 3$  from infinity to  $a$ , skips  $a$  because of the first pole, continues down to  $b$ , skips  $b$  at the second pole, then traces out the rest of the line to negative infinity as the third pole is approached from below. On the other side of the third pole the entire line is retraced from negative infinity to positive infinity without any skips. That makes it obvious why  $a$  and  $b$  each have exactly one inverse image in the  $(x, y)$ -plane.

$F$  is not birational, because it is generically two to one. Throughout the remainder of this section, let  $k = \mathbb{R}$ . To begin the exploration of the field extension  $k(P, Q) \subset k(x, y)$ , rewrite the parametrization above in terms of  $f$  and  $h$ , using the relations  $P = c$  and  $P = f + h$  to obtain

$$x = f(h+1)(f-h-h^2)^{-2}$$

$$y = (f-h-h^2)^2(f-h^2)f^{-2},$$

which are identities in  $k(x, y)$  (and so  $k(x, y) = k(f, h)$ ). It follows that  $xy = (h+1)(f-h^2)/f$ ,  $t = xy - 1 = [(h+1)(f-h^2)-f]/f = [fh-h^2-h^3]/f = (h/f)[f-h(h+1)]$ ,  $q = -t^2 - 6th(h+1) = -h^2f^{-2}\{[f-h(h+1)]^2 + 6(h+1)[f-h(h+1)]f\}$ . In fact,

$$q = -h^4(h+1)^2/f^2 + [2h^3(h+1) + 6h^3(h+1)^2]/f + [-h^2 - 6h^2(h+1)]$$

$$= -h^4(h+1)^2/f^2 + h^3(h+1)(6h+8)/f - h^2(6h+7).$$

Using that equation, the definition  $Q = q - u(f, h)$ , and equation 1 for  $u$  one can express  $Q$  in terms of  $f$  and  $h$  alone. Clearing denominators  $f^2Q = f^2q - f^2u$ , or,

arranged by powers of  $f$ ,

$$\begin{aligned} f^2Q &= -h^4(h+1)^2 \\ &\quad + f[h^3(h+1)(6h+8)] \\ &\quad + f^2[-h^2(6h+7) - 91h^2 - 69h^3 - (75/4)h^4] \\ &\quad + f^3[-170h - 195h^2 - 75h^3]. \end{aligned}$$

Now substitute  $P-h$  for  $f$  and collect in powers of  $h$  to obtain a polynomial relation

$$(2) \quad (197/4)h^6 + \cdots + (2PQ - 170P^3)h - P^2Q = 0.$$

Let  $R(T)$  be the corresponding polynomial in  $T$  with root  $h$ .  $R(T)$  is explicitly written out in full in the Appendix (section 4). It is clear, even without an explicit formula, that the coefficient of each power of  $T$  is a polynomial in  $P$  and  $Q$  with rational coefficients, and has total degree in  $P$  and  $Q$  at most 3. Since the leading coefficient of  $R(T)$  is a real constant, the fact that  $R(h) = 0$  shows that  $h$  is integral over  $k[P, Q]$ .

Let  $m(T)$  be the polynomial in  $k[P, Q][T]$ ,  $T$  an indeterminate, which has leading coefficient 1 and satisfies  $m(h) = 0$  in  $k[x, y]$ , and which is of minimal degree. Clearly  $m$  is irreducible in  $k[P, Q][T] = k[P, Q, T]$  and hence by the Gauss Lemma, in  $k(P, Q)[T]$ . That implies that  $m$  is also of minimal degree over  $k(P, Q)$ , that  $m$  divides any polynomial in  $k[P, Q][T]$  with  $h$  as a root, and that  $m$  is unique.

Note the following  $k$ -linear field inclusions

$$k(P, Q) \subset k(P, Q)(h) = k(f, h) = k(x, y).$$

Next consider the  $k$ -algebra homomorphisms

$$k[P, Q] \subset k[P, Q][h] \subseteq k[x, y]$$

and the corresponding regular maps of affine real algebraic varieties

$$k^2 \rightarrow \text{Zeroset}(m) \rightarrow k^2$$

with the first map sending  $(x, y)$  to  $(P(x, y), Q(x, y), h(x, y))$  and the second the projection onto the first two components. The first map is birational ( $k(P, Q)(h) = k(x, y)$ ) and the second is finite (topologically proper with an overall bound on the number of inverse images of points in the codomain) by integrality. Incidentally, that shows that the inclusion  $k[P, Q][h] \subseteq k[x, y]$  is actually strict, since  $F$  is not topologically proper.

If we fix any point  $w$  in the  $(P, Q)$ -plane, we may consider  $m$  as a polynomial in  $T$  with real coefficients and real roots that determine the points projecting onto  $w$  under the second map. So we call the algebraic surface  $m = 0$  the variety of real roots of  $m$  and if  $m(w, r) = 0$  we say that  $r$  is a root of  $m$  over  $w$ . Note that for any point of the  $(x, y)$ -plane,  $h(x, y)$  is a real root of  $m$  over  $w = F(x, y)$ .

**Lemma 1.** *For generic  $w$ ,  $m$  has exactly two real roots over  $w$ , they are simple and distinct,  $w$  has exactly two inverse images  $v$  and  $v'$  under  $F$ , and the real roots of  $m$  over  $w$  are  $r = h(v)$  and  $r' = h(v')$ .*

*Proof.* Take a bi-regular isomorphism from a Zariski open subset  $O$  of the  $(x, y)$ -plane to a Zariski open subset of the variety of real roots of  $m$ . The image is a nonsingular surface  $S$ . In the usual (strong) topology it has a finite number of connected components that are open subsets of  $S$  and of the variety of real roots. Take the union of i) the image of the complement of  $O$  under  $F$ , ii) the projection

of the complement of  $S$ , and iii)  $A(F)$ , the asymptotic variety of  $F$ . From the Tarski-Seidenberg projection property and other basic tools of real semi-algebraic geometry, the union is semi-algebraic of maximum dimension 1. Take  $w$  in the complement of the Zariski closure of that union. Any root that lies over  $w$  is a nonsingular point of the variety of real roots (by construction), and so is a simple, not multiple, real root. There are exactly two points, say  $v$  and  $v'$ , that map to  $w$  under  $F$ . Their images under  $h$ ,  $r$  and  $r'$ , lie over  $w$ . By construction  $(w, r)$  and  $(w, r')$  lie in  $S$ , and  $v$  and  $v'$  lie in  $O$ . Hence  $r$  and  $r'$  are distinct. No point in the complement of  $O$  can map to  $(w, r)$  or  $(w, r')$  (by construction), and  $v$  and  $v'$  are the only points in  $O$  that do so.  $\square$

**Corollary 2.** *The  $T$ -degree of  $m$  is even.*

*Proof.* The complex roots over  $w$  that are not real occur in complex conjugate pairs.  $\square$

Let  $m_0$  be the term of  $m(T)$  of degree 0 in  $T$ . Clearly,  $m_0 \in k[P, Q]$  is not the zero polynomial. Since  $m(T)$  divides  $R(T)$ ,  $m_0$  divides  $P^2Q$ . Since the  $T$ -degree of  $m$  is even,  $m_0(w)$  is the product of all the roots, real and complex, of  $m$  over  $w$ , for any  $w$  in the  $(P, Q)$ -plane.

**Proposition 3.**  *$m_0$  is a positive constant multiple of  $-P^2Q$ .*

*Proof.*  $F(1, 0) = (0, -1)$  and  $h(1, 0) = 0$ , so  $m_0(0, -1) = 0$ . This shows that  $P$  must divide  $m_0$ , for otherwise  $m_0(0, -1)$  would be nonzero. Next,  $F(1, 1) = (1, 0)$  and  $h(1, 1) = 0$ , so  $m_0(1, 0) = 0$ . So  $Q$  divides  $m_0$ . Next, consider the union of the vertical lines  $P = c$  in the  $(P, Q)$ -plane, for  $2 < c < 4$ . At least one such line must contain a point  $w$  that is generic in the sense of Lemma 1, for otherwise there would be an open set of nongeneric points. Choose such a  $c$ , and note that all but finitely many points of the line  $P = c$  are generic. The level set  $P = c$  in the  $(x, y)$ -plane has the rational parametrization by  $h$  already described, with a pole at  $h = c$ , at which  $Q$  tends to  $-\infty$ . Take a point  $w = (c, d)$  with  $d$  negative and sufficiently large. Then  $w$  will be generic and its two inverse images under  $F$  will have values of  $h$  that approach  $c$  as  $d$  tends to  $-\infty$ , one value of  $h$  less than  $c$  and the other greater. The product of all the roots of  $m$  over  $w$  will be positive, since the nonreal roots occur as conjugate pairs. Since  $P$  is positive and  $Q$  negative, the numerical coefficient of  $m_0$  must be negative, regardless of whether  $m_0$  is exactly divisible by  $P$  or by  $P^2$ . Finally, make a similar argument for a suitable line  $P = c$ , with  $-4 < c < -2$ . Consideration of signs shows that  $m_0$  must be divisible by  $P^2$ , which yields the desired conclusion.  $\square$

**Corollary 4.** *The  $T$ -degree of  $m$  is not 2.*

*Proof.* If  $m$  has degree 2 and  $w$  is generic, then  $m_0(w)$  is exactly the product of the two real roots of  $m$  over  $w$ . But for the last two examples considered in the previous proof, the product tends to  $c^2$  as  $h$  tends to  $c$ , whereas  $m_0(w)$  is unbounded.  $\square$

**Proposition 5.**  $R(T) = (197/4)m(T)$ .

*Proof.* As  $m(T)$  is a nonconstant divisor of  $R(T)$  of even  $T$ -degree not equal to 2, it remains only to show that the degree of  $m$  in  $T$  is not 4. Suppose to the contrary that  $(m_4T^4 + m_3T^3 + m_2T^2 + m_1T + m_0)(d_2T^2 + d_1T + d_0) = R(T)$ , where the first factor is  $m(T)$ , the second is a polynomial  $D(T)$  of degree 2 in  $T$ , and all the coefficients

shown are in  $k[P, Q]$ . Equating leading and constant terms on both sides, one finds that  $m_4 = 1$ ,  $d_4 = 197/4$  and  $d_0$  is a positive constant. The coefficient  $d_1$  must also be constant. For if not,  $j = \deg^t(d_1) > 0$ , where  $\deg^t$  temporarily denotes the total degree in  $P$  and  $Q$ . As noted earlier, that  $\deg^t$  is at most 3 for every coefficient of  $R(T)$ . Starting with  $\deg^t(m_0) = 3$  and equating in turn terms of  $T$ -degree 1 through 4 on both sides of the equation assumed for  $R(T)$ , one readily finds that  $\deg^t(m_4) = 3+4j$ . But  $m_4 = 1$ , a contradiction. Thus  $D(T)$  has constant coefficients. Next, set  $P = 0$  in  $R(T)$ , obtaining  $(197/4)T^6 + 104T^5 + 63T^4 - QT^2$ . That result can be found easily by setting  $f = -h$  in the rational equation for  $Q$  in terms of  $f$  and  $h$ . Further setting  $Q = 0$ , one finds that the resulting polynomial in  $T$  alone factors as  $T^4((197/4)T^2 + 104T + 63)$ . Clearly  $D$  must be exactly the quadratic factor shown. But if  $D(T)$  divides  $R(T)$ , setting  $P = 0$  implies it must also divide  $-QT^2$ , which is absurd. That contradiction shows that the original assumption to the contrary, that  $m$  has  $T$ -degree 4, is false.  $\square$

**Corollary 6.** *The field extension  $k(P, Q) \subset k(x, y)$  is of degree 6.*

*Proof.* Clear.  $\square$

### 2.3. Automorphisms of the extension.

This section examines automorphisms of the extension  $k(P, Q) \subset k(x, y)$ ; that is, field automorphisms of  $k(x, y)$  that fix every element of  $k(P, Q)$ . Again, assume throughout that  $k = \mathbb{R}$ .

First, consider the geometry over  $\mathbb{R}$ . Let  $Z = F^{-1}(A(F))$ . For any  $(x, y) \notin Z$  there is a unique different point  $(x', y') \notin Z$  with the same image under  $F$ . There is an obvious Klein four group  $\{e, \sigma, \sigma', \tau\}$  of  $F$ -preserving transformations of the complement of  $Z$ , where  $e$  is the identity map,  $\sigma$  interchanges inverse images over points that lie on one side of  $A(F)$ ,  $\sigma'$  is defined similarly for the other side of  $A(F)$ , and  $\tau = \sigma\sigma' = \sigma'\sigma$  leaves no point invariant, always interchanging  $(x, y)$  and  $(x', y')$ . These geometric involutions are Nash diffeomorphisms, that is, they are real analytic and semi-algebraic. Except for the identity map  $e$ , these transformations cannot be even locally extended analytically to any point  $z \in Z$ . For otherwise,  $z$  would be a fixed point, and since  $F$  is a local diffeomorphism the map would be the identity map on a neighborhood of  $z$ , thus over both sides of  $A(F)$ .

Next, the algebra. Suppose  $\varphi$  is an automorphism of  $k(x, y)$  that is not the identity but fixes every element of  $k(P, Q)$ . If  $\varphi$  preserves  $h$ , then it also preserves  $x$  and  $y$ , since they are rational functions of  $P$  and  $h$ , namely

$$(3) \quad x = \frac{(P-h)(h+1)}{(P-2h-h^2)^2} \text{ and } y = \frac{(P-2h-h^2)^2(P-h-h^2)}{(P-h)^2}.$$

So  $\varphi(h) = h' \neq h$  and, furthermore, by the identity principle for rational functions over  $\mathbb{R}$ , they cannot be equal on any nonempty open subset of the  $(x, y)$ -plane on which  $h'$  is defined. Since  $\varphi$  preserves both components of  $F$ , the fact that its geometric realization cannot be the identity even locally means that it must be  $\tau$  (see above) wherever both are defined. That implies that there can be at most one such a nonidentity automorphism  $\varphi$ . If it exists, then the rational function  $h'$  is analytic at any point  $(x, y) \notin Z$ , since  $h'(x, y) = h(x', y') = h(\tau(x, y))$ .

That reduces the question of the existence of  $\varphi$  to the following one. Is  $h'$ , a well defined real analytic function on the complement of  $Z$  in the  $(x, y)$ -plane, in fact a real rational function?

**Lemma 7.** *There are three component curves of the level set  $P = 0$  on which  $h$  is nonconstant and vanishes nowhere. On those curves  $Q = Q(h) = h^2((197/4)h^2 + 104h + 63)$  and is everywhere positive. There is one point of  $Z = F^{-1}(A(F))$  on the three curves. At all of their other points,  $h'$  satisfies both  $h' \neq h$  and  $Q(h') = Q(h)$ .*

*Proof.* Set  $P = 0$  in equation 3. The resulting rational functions  $x(h), y(h)$  are defined everywhere except at  $h = -2$  and  $h = 0$ . That yields three curves parametrized by  $h$ . Since  $h(x(P, h), y(P, h))$  simplifies to  $h$  for the rational functions in equation 3, it follows that  $h(x(h), y(h)) = h$ , and hence  $h$  assumes every real value exactly once on these curves, except that  $-2$  and  $0$  are never assumed. Since every level set  $P = c$  is a finite disjoint union of closed connected smooth curves unbounded at both ends, each of the three curves is a connected component of  $P = 0$ .

Next, set  $P = 0$  in equation 2 of section 2.2, which is the relation  $R(h) = 0$ , where  $R(T)$  is that section's minimal degree, but not monic, polynomial with root  $h$ . The result, which in essence already appeared in the proof of Proposition 5, is  $(197/4)h^6 + 104h^5 + 63h^4 - Qh^2 = 0$ . On the curves,  $h \neq 0$ , so the claimed formula for  $Q$  follows. Furthermore,  $Q$  is positive there, since  $(197/4)h^2 + 104h + 63$  has negative discriminant.

So  $F$  maps points on the three curves to the positive  $Q$ -axis. Routine calculation of the derivative of  $Q$  shows that  $Q$  is monotonic decreasing for  $h < 0$  and monotonic increasing for  $h > 0$ . Considering the graph of  $Q$ , one concludes that every positive real is the value of  $Q$  exactly twice, for nonzero values of  $h$  of opposite signs. Those values of  $h$  all correspond to unique points of the curves, except  $h = -2$ . As  $Q(-2) = 208$ , the point  $(0, 208)$ , which is the only point of  $A(F)$  on the positive  $Q$ -axis, has as its unique inverse under  $F$  the point  $(x(h'), y(h'))$ , where  $h'$  is the positive real satisfying  $Q(h') = 208$ .  $\square$

**Remark.** To clarify, there are two additional component curves of the level set  $P = 0$ . On them  $h = 0$  identically. They have a rational parametrization by  $t$  with a pole at  $t = 0$  and  $Q = -t^2$  is everywhere negative on them. They contain no point of  $Z$ ,  $t' = -t$ , and  $h' = 0$ .

**Lemma 8.** *Let  $h'$  be any real rational function of  $h$  that satisfies  $Q(h') = Q(h)$  for infinitely many values of  $h \in \mathbb{R}$ . Then  $h' = h$ .*

*Proof.* Suppose  $h' = a/b$  for polynomials  $a, b \in \mathbb{R}[h]$ , of respective degrees  $r, s$ , with no common divisor. From  $Q(h') = Q(h)$  one obtains

$$(4) \quad a^2((197/4)a^2 + 104ab + 63b^2) = b^4h^2((197/4)h^2 + 104h + 63),$$

a polynomial equality that is true for all real  $h$ . Since  $a/b$  tends to  $\infty$  as  $h$  does,  $r > s$ . Counting degrees  $4r = 4s + 4$ , so  $r = s + 1$ . The factor in parentheses on the left is quadratic and homogeneous in  $a$  and  $b$  and has negative discriminant, so it is zero for real  $a$  and  $b$  only if both are zero. But that cannot occur for any real  $h$ , for then  $a$  and  $b$  would have a common root, hence a common divisor. Thus that factor has only complex roots. It follows that  $h^2$  divides  $a^2$ , and so  $h$  divides  $a$ . That means that  $a$  and  $bh$  share a root, each having  $h$  as a factor. As the quadratic



factor in parentheses on the right is not zero for any real  $h$ , any further real roots (at  $h = 0$  or not) in the two sides of equation 4 would be shared by  $a$  and  $b$ . Again, that is not possible, and therefore  $a$  has no further real roots and all the roots of  $b$  are complex. In particular  $s$ , the degree of  $b$ , must be even. No complex root of  $b$  can be a root of  $a$ , as that would imply a common real irreducible quadratic factor. So it must be a root of the parenthetical factor on the left. Counting roots with multiplicities,  $2r = 2s + 2 \geq 4s$ , so  $s \leq 1$ . Since  $s$  is even, it must be 0, and so  $h' = \lambda h$  for a nonzero  $\lambda \in \mathbb{R}$ . Then for any fixed  $h \neq 0$ ,  $Q(\lambda^i h) = Q(h)$  is independent of  $i > 0$ , and so  $\lambda$  has absolute value 1. It cannot be  $-1$ , because  $Q(h) - Q(-h) = 208h^3$ .  $\square$

**Proposition 9.** *The group of automorphisms of the field extension  $k(P, Q) \subset k(x, y)$  is trivial. In particular, the extension is not Galois.*

*Proof.* If the group contains a nontrivial automorphism  $\varphi$ , then  $h' = \varphi(h) = h(x', y')$  (see above) belongs to  $\mathbb{R}(x, y) = \mathbb{R}(P, h)$ . As  $\mathbb{R}(P, h)$  is a rational function field in two algebraically independent elements over  $\mathbb{R}$ , the restriction of  $h'$  to the level set  $P = 0$  must either be generically undefined (uncanceled  $P$  in the denominator) or a rational function of  $h$ . The first case is ruled out by Lemma 7, which also contradicts Lemma 8 in the second case.  $\square$

#### 2.4. All Pinchuk maps.

From a geometric point of view, any two different Pinchuk maps are very closely related. More specifically, if  $F_1 = (P, Q_1)$  and  $F_2 = (P, Q_2)$  are Pinchuk maps then they have the same first component,  $P$ , and their second components satisfy  $Q_2 = Q_1 + S(P)$  for a polynomial  $S$  in one variable with real coefficients. As maps of  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , therefore, they differ only by a triangular polynomial automorphism of the image plane. In effect, all Pinchuk maps share the same geometry. Indeed, the set  $Z = F^{-1}(A(F))$  and the real analytic function  $h'$  of the previous section 2.3 are exactly the same for all Pinchuk maps. Moreover, all Pinchuk maps are generically two to one, and their asymptotic varieties have algebraically isomorphic embeddings in the image plane.

Remark. In [12] Janusz Gwoździejewicz studied a Pinchuk map of total degree 40 and noted the single point in the Zariski closure of the asymptotic variety not on the variety itself. He first brought to my attention the polynomial relation between different Pinchuk maps in an informal communication in 2009.. An algebraically oriented proof, along lines suggested by Arno van den Essen, can be found in [6].

Let  $F$  be the same Pinchuk map as before. It is defined over  $\mathbb{Q}$ . In fact, not only do  $P$  and  $Q$  have rational coefficients, but so do  $h$  and all terms of the minimal polynomial  $m$  for  $h$ . Let  $k$  be any subfield of  $\mathbb{R}$ , including the possibilities  $k = \mathbb{Q}$  and  $k = \mathbb{R}$ . Then the powers  $h^i$  for  $i = 0, \dots, 5$  form a basis for  $k[P, Q][h]$  as a free module over  $k[P, Q]$  and for  $k(x, y)$  as a vector space over  $k(P, Q)$ . So the field extension  $k(F) = k(P, Q) \subset k(X) = k(x, y)$  is of degree 6.

**Proposition 10.** *Let  $F_1 = (P, Q_1)$  and  $F_2 = (P, Q_2)$  be Pinchuk maps and let  $Q_2 = Q_1 + S(P)$  for a polynomial  $S$  in one variable with real coefficients. Then  $S$  is uniquely determined and its coefficients belong to any subfield of  $\mathbb{R}$  that contains the coefficients of  $Q_1$  and  $Q_2$ .*

*Proof.*  $P$  is transcendental over  $\mathbb{R}$ , so  $S$  is unique. Let  $k$  be the subfield of  $\mathbb{R}$  in question. Let  $c \in \mathbb{Q}$  with  $c \neq 0$  and  $c \neq -1$ . Choose  $h \in \mathbb{Q}$  that is not a pole of the previously described rational parametrization  $x(h), y(h)$  of the level set  $P = c$ . Since both  $x(h)$  and  $y(h)$  have formulas in  $\mathbb{Q}(h, c)$ , the real number  $S(c) = Q_2(x(h), y(h)) - Q_1(x(h), y(h))$  actually is in  $k$ . The coefficients of  $S$  can be reconstructed, using rational arithmetic, from its values at any  $j > \deg S$  such points  $c$ , and so are in  $k$ .  $\square$

**Corollary 11.** *Let  $F_1$  and  $F_2$  be Pinchuk maps and let  $k$  be  $\mathbb{R}$  or any subfield of  $\mathbb{R}$  over which both maps are defined. Then  $k(F_1) \subset k(X)$  and  $k(F_2) \subset k(X)$  are one and the same field extension.*

*Proof.* Since  $S$  has coefficients in  $k$ , the relation  $Q_2 = Q_1 + S(P)$  implies that  $k(F_1) = k(P, Q_1) = k(P, Q_2) = k(F_2)$ .  $\square$

**Theorem 2.1.** *Let  $F$  be any Pinchuk map and let  $k$  be  $\mathbb{R}$  or any subfield of  $\mathbb{R}$  containing the coefficients of  $F$ . Although  $F$  is generically two to one as a polynomial map of  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , the degree of the associated extension of function fields over  $k$  is six. Furthermore, the extension has no automorphisms other than the identity, and so, in particular, it is not Galois.*

*Proof.* The conclusions have already been drawn for the earlier specific Pinchuk map  $F = (P, Q)$  and for  $k = \mathbb{R}$  (Corollary 6 and Proposition 9). Both Pinchuk maps have the same function field extension over  $k$ , so it has degree six. And any nontrivial automorphism defined over  $k$  defines one over  $\mathbb{R}$ , since the  $\mathbb{R}$ -linearly extended automorphism preserves  $P$ ,  $Q$ , and  $\mathbb{R}$ .  $\square$

### 3. REAL JACOBIAN CONJECTURES

The Strong Real Jacobian Conjecture (SRJC), as formulated in the introduction, asserted that a polynomial map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , has a real analytic inverse if its Jacobian determinant,  $j(F)$ , vanishes nowhere on  $\mathbb{R}^n$ . It was refuted by the Pinchuk counterexamples, so only special cases are continuing subjects of inquiry. Both the hypothesis concerning  $j(F)$  and the conclusion that a real analytic inverse exists can be restated in various equivalent ways. Principally, the former is equivalent to the assertion that  $F$  is locally diffeomorphic or locally real bianalytic, and the latter to the assertion that  $F$  is injective (one-to-one) or bijective (one-to-one and onto) or a homeomorphism or a diffeomorphism. These are all obvious, except for the key result that injectivity, also called univalence, implies bijectivity for maps of  $\mathbb{R}^n$  to itself that are polynomial or, more generally, rational and defined on all of  $\mathbb{R}^n$  [13]. That result does not generalize to semi-algebraic maps of  $\mathbb{R}^n$  to itself [14]. To avoid any possible confusion, an everywhere defined real rational map has components that belong to  $\mathbb{R}(x_1, \dots, x_n)$ , each of which can be written as a fraction with a polynomial numerator and a nowhere vanishing polynomial denominator. In addition to their global and local properties in the strong (Euclidean) topology, such maps are continuous in the Zariski topology. Let the (false) rational real Jacobian conjecture (RRJC)) be the extension of the SRJC to everywhere defined rational maps. Clearly any global univalence theorems [15] that apply to locally diffeomorphic maps  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  yield (true) special cases of the conjecture. Properness suffices, and related topological considerations play a role below. But the focus of this article is on results or conjectures that require the polynomial or

rational character of a map and involve properties of the associated algebraic field extension .

Remark. The term 'rational real Jacobian conjecture' and the acronym RRJC are not standard terminology. Even if rational maps are allowed, the conjecture is usually called the real Jacobian conjecture or even just the Jacobian conjecture, despite considerable ambiguity. And the term 'rational' is used with various shades of meaning. In some work on the two dimensional complex JC (e.g., [16]), a rational polynomial is a polynomial in two complex variables whose generic fiber is a rational complex curve. And Vitushkin [17] has presented  $F = (x^2y^6 + 2xy^2, xy^3 + 1/y)$  as a sort of rational counterexample to the JC.  $F$  has constant Jacobian determinant  $j(F) = -2$  and  $F(-3, -1) = F(1, 1) = (3, 2)$  So  $F$  is not injective when considered as a rational map of  $\mathbb{R}^2$  to itself. But  $F$  is also not defined everywhere on  $\mathbb{R}^2$ .

In the RRJC context, the distinction between nonzero constant and nowhere vanishing Jacobian determinants is not as critical as it may seem. If  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies the hypotheses, let  $x \in \mathbb{R}^n, z \in \mathbb{R}$  and define  $F^+ : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  by  $F^+(x, z) = (F(x), z/(j(F)(x)))$ . Then  $F^+$  also satisfies the hypotheses,  $j(F^+) = 1$ , and  $F^+$  is injective if, and only if,  $F$  is injective. As pointed out in [18], choosing Pinchuk maps for  $F$  yields counterexamples to the RRJC in dimension  $n = 3$  with nonzero constant Jacobian determinant.

Note that all three conjectures are true in the dimension  $n = 1$  case  $f : \mathbb{R} \rightarrow \mathbb{R}$ . In the JC case,  $f$  is of degree 1. In the SRJC case,  $f$  is proper, since any nonconstant polynomial becomes infinite when its argument does. In the RRJC case,  $f$  is monotone increasing or decreasing, hence injective, thus surjective, so unbounded above and below, and therefore proper.

Let  $F = (f_1, \dots, f_n)$  be a real rational map of  $n$  real variables  $x_1, \dots, x_n$  and  $k$  a subfield of  $\mathbb{R}$  such that each component  $f_i$  belongs to  $k(X) = k(x_1, \dots, x_n)$ . Whether defined on all of  $\mathbb{R}^n$  or not, if  $j(F)$  is not identically zero, then  $k(F) = k(f_1, \dots, f_n) \subseteq k(X)$  is an algebraic field extension of finite degree. The reasons are the same as in the polynomial case, and the following standard lemma shows they are true in broader contexts having nothing to do with  $\mathbb{R}$ .

**Lemma 12.** *Let  $k$  be a field of characteristic zero and suppose that  $f_1, \dots, f_n \in k(x_1, \dots, x_n)$  are algebraically dependent over  $k$ . Let  $j(F) \in k(x_1, \dots, x_n)$  be the Jacobian determinant of  $F = (f_1, \dots, f_n)$ . Then  $j(F) = 0$ .*

*Proof.* Suppose, to the contrary, that  $h(f_1, \dots, f_n) = 0$  for a nonzero polynomial  $h \in k[y_1, \dots, y_n]$ , but  $j(F) \neq 0$ . Put  $K = k(x_1, \dots, x_n)$  and observe that  $J(F)$ , the Jacobian matrix of  $F$ , is an invertible matrix with entries in the field  $K$ . From the chain rule  $v \cdot J(F) = 0$ , where  $v$  is the row vector with components  $v_i = (\partial h / \partial y_i)(f_1, \dots, f_n) \in K$ . It follows that  $v = 0$ , and so each first order partial derivative of  $h$  is either the zero polynomial or it defines a new relation of algebraic dependence. By induction, the same is true for partials of all orders. But since  $h$  is a polynomial, at least one partial of some order is a nonzero constant, yielding a contradiction.  $\square$

So both the Galois and birational cases of the RRJC are meaningful. The Galois case of the standard JC is settled and true over any field of characteristic zero. It states that a polynomial map with constant nonzero Jacobian determinant and a Galois field extension has a polynomial inverse. A fairly recent proof [2, Thm. 2.2.16] is accompanied by historical notes [2, p. 60] on earlier proofs and partial

results. Of course, the existence of a polynomial inverse implies the triviality of the field extension, so the theorem has no concrete examples. In contrast, in the SRJC and RRJC contexts, the existence of an inverse does not imply the field extension is Galois, much less birational. For instance, if  $y = f(x) = x^3 + x$ , the field extension  $\mathbb{R}(y) \subset \mathbb{R}(x)$  is neither.

The following are basic properties of an everywhere defined real rational map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $j(F)$  vanishing nowhere. It is a local diffeomorphism, hence an open map. Let  $x \in \mathbb{R}^n$  and  $y = F(x) \in \mathbb{R}^n$  and define  $m(x)$  to be the number of inverse images of  $y$  under  $F$ , potentially allowing  $+\infty$  as a possible value. Since  $F$  is open,  $m(x') \geq m(x)$  for  $x' \in \mathbb{R}^n$  in a neighborhood of  $x$ . So if  $A \subseteq \mathbb{R}^n$ , the maximum value of  $m$  on  $A$  is also the maximum value of  $m$  on its topological closure  $\bar{A}$ . Let  $t \in \mathbb{R}(X)$  be a primitive element (generator) for the field extension  $\mathbb{R}(F) \subseteq \mathbb{R}(X)$ , and suppose it satisfies an equation of minimal degree  $d$  in  $t$  over  $\mathbb{R}[F]$ . Temporarily define a Zariski open subset  $U$  of  $\mathbb{R}^n$  by declaring  $x \in U$  if  $y = F(x)$  is not a zero of the leading coefficient of the equation for  $t$  and  $x$  is not a zero of a selected specific common denominator for  $t$  and the coefficients of the expressions for the components of  $F$  as polynomials in  $t$ . If  $x \in U$ , then  $t(x)$  can have at most  $d$  values and so  $m(x)$  is also not greater than  $d$ . Let  $N \leq d$  be the maximum value of  $m(x)$  on  $U$ . Because  $\bar{U} = \mathbb{R}^n$ ,  $N$  is in fact the maximum value of  $m(x)$  for all  $x \in \mathbb{R}^n$ . That shows not only that  $F$  is quasifinite, meaning that the inverse image of any point in the codomain is a finite set, but also that the degree of the field extension  $\mathbb{R}(F) \subseteq \mathbb{R}(X)$  is a global bound on the size of those sets. All subsets of  $\mathbb{R}^n$  that can be described in the first order logic of ordered fields are semi-algebraic. The description can include real constant symbols (coefficients, values, etc.) and quantification over real variables (but not over subsets, functions or natural numbers); results for any dimension  $n > 0$  and involving polynomials of arbitrary degrees follow from schemas specifying first order descriptions for any fixed choice of the natural number parameters. As a first application of that principle, the  $N$  subsets of the domain  $\mathbb{R}^n$  on which  $m(x)$  has a specified numeric value in the range  $1, \dots, N$ , and the  $N + 1$  subsets of the codomain  $\mathbb{R}^n$  on which  $y$  has a specified number of inverse images in the range  $0, \dots, N$ , are all semi-algebraic. The set of points  $y$  in the codomain at which  $F$  is proper is readily verified to be the open set of points at which the number of inverse images of  $y$  is locally constant. That set contains all points with  $N$  inverse images and has an  $\epsilon$ -ball first order description. Its complement  $A(F)$ , the asymptotic variety of  $F$ , is therefore closed semi-algebraic and the inclusion  $A(F) \subset \mathbb{R}^n$  is strict.  $A(F)$  is the union for  $i = 0, \dots, N - 1$  of the semi-algebraic sets consisting of points  $y$  in the codomain at which  $F$  is not proper and for which  $y$  has exactly  $i$  inverse images. At an interior point  $y$  of one of these sets  $F$  would be proper, contradicting  $y \in A(F)$ . Thus each such set has empty interior, hence is of dimension less than  $n$ . Consequently  $\dim A(F) < n$ . It follows that the complement of  $A(F)$  is a finite union of disjoint connected open semi-algebraic subsets of  $\mathbb{R}^n$  on each of which the number of inverse images of points is a constant, with possibly differing constants for different connected components. If  $U$  is such a connected component, then  $F^{-1}(U)$  is also open and semi-algebraic. If it is not empty, let  $V$  be one of its finitely many connected components. Since  $V$  is an open and closed subset of  $F^{-1}(U)$ , the map  $V \rightarrow U$  induced by  $F$  is a proper local homeomorphism of connected, locally compact, and locally arcwise connected spaces and hence it is a covering map. Such a map is surjective, so all of  $U$  is contained in  $F(\mathbb{R}^n)$ .  $V$

must be exactly one of the finitely many connected components of the open semi-algebraic set  $\mathbb{R}^n \setminus F^{-1}(A(F))$ , since it is closed in that subset as one element of a finite cover by total spaces of covering maps. Speaking informally, this presents a view of  $F$  as a finite collection of  $n$ -dimensional covering maps, of possibly different degrees, glued together along semi-algebraic sets of positive codimension to form  $\mathbb{R}^n$  at the total space level, whose base spaces, which may sometimes coincide for different total spaces, are similarly glued together to form  $F(\mathbb{R}^n)$ . While  $F(\mathbb{R}^n)$  is open and connected, it would not be dense in  $\mathbb{R}^n$  if  $F^{-1}(U) = \emptyset$  for some connected component  $U$  of  $\mathbb{R}^n \setminus A(F)$ , a possibility that has not been ruled out.  $F(\mathbb{R}^n) \cap A(F)$  is in general neither empty nor all of  $A(F)$ , a behavior exhibited by any Pinchuk map.

**Remark.** There is an extensive body of work by Zbigniew Jelonek defining, investigating, or related to the concept of the set of points at which a polynomial map is not proper. In [11] he covers and sharpens the just described facts, at least for polynomial maps. As one result, he shows that for a nonconstant polynomial map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , where  $n$  and  $m$  are any positive integers and no other conditions are imposed, the set  $A(F)$  of points at which  $F$  is not proper is  $\mathbb{R}$ -uniruled. By that he means that for any  $a \in A(F)$  there is a nonconstant polynomial map  $g : \mathbb{R} \rightarrow \mathbb{R}^m$  (a polynomial curve) such that  $g(0) = a$  and  $g(t) \in A(F)$  for all  $t \in \mathbb{R}$ . That in turn implies that every connected component of  $A(F)$  is unbounded and has positive dimension. In the same article, Jelonek explicitly considers the SRJC and shows, using topological methods, that  $F$  has an inverse (and hence  $A(F) = \emptyset$ ) if  $A(F)$  has codimension three or higher. That and related results will be reconsidered below in the RRJC context.

*Example 1.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the real rational map given by  $y = 1/(x^2 + 1)$ . The image  $f(\mathbb{R})$  is the half open interval  $0 < y \leq 1$ . The point  $y = 0$  is the only point at which  $f$  is not proper. So  $A(f) = \{0\}$  is of dimension 0 and not uniruled.

### 3.1. Points of definition.

This section states a number of standard definitions and assembles some associated technical results for later use.

Let  $k$  be any field. Characteristic zero is not assumed. It is not technically correct to speak of *the coefficients* of a rational function, since it is an equivalence class of (numerator, denominator) pairs. A rational function is said to be *defined over  $k$*  if it has a representative pair with coefficients in  $k$ . And a rational function of  $n$  variables defined over  $k$  is said to be *defined at a point  $x \in k^n$*  if the denominator can also be chosen so that it is not zero at  $x$ . Eliminating common factors of the numerator and denominator yields a *reduced fraction*, and unique factorization shows that all reduced fraction representations of a given rational function have the same numerator and the same denominator, up to multiplication by nonzero elements of  $k$ ; such a denominator is zero exactly at the points at which the function is not defined. Any reduced fraction for  $f \in k(X)$  can be used to determine if  $f$  is defined at a point  $x \in k^n$ , and if so, to compute its value  $f(x)$ ; this also applies if  $x$  is allowed to have coordinates in a commutative  $k$ -algebra.

Let  $K$  be any field containing  $k$  as a subfield.

**Lemma 13.** *Let  $a, b, c \in K[X]$  satisfy  $a = bc \neq 0$ . If any two of them belong to  $k[X]$ , so does the third.*

*Proof.* Clear if  $b, c \in k[X]$ . Suppose, without loss of generality, that  $a, c \in k[X]$ . Choose a term order for  $x_1, \dots, x_n$ . That is, choose a total order on monomials in  $x_1, \dots, x_n$  compatible with multiplication. Comparison of leading terms reveals that  $b$  has as leading term  $t$  the product of a monomial and a coefficient in  $k$ . Let  $a' = a - tc = (b - t)c$ . If  $a' = 0$ , then  $b = t \in k[X]$ . If not, conclude by descending induction on the maximum order of terms in  $b$ .  $\square$

**Lemma 14.** *If  $K/k$  is purely transcendental and  $p \in k[X]$  is irreducible, then  $p$  remains irreducible in  $K[X]$ .*

*Proof.* If  $p = ab$ , then the factorization actually takes place in  $k[X]$ . To prove that, first, assume a simple transcendental extension  $K = k(t)$ . Then both factors must have degree zero in  $t$ . Next, induction handles finite transcendence degree. Finally, a counterexample could only involve finitely many elements of a transcendence base.  $\square$

**Lemma 15.** *If  $K/k$  is algebraic and  $p, q \in k[X]$  are relatively prime, then they remain relatively prime in  $K[X]$ .*

Remark. The following proof was privately communicated by Hyman Bass.

*Proof.* Let  $A = k[X]$  and  $B = K[X]$ . As an extension of commutative rings,  $B/A$  is integral, because  $B$  is generated over  $A$  by the elements of  $K$ , which are algebraic over  $k$  and hence integral over  $A$ . Let  $J_k$  ( $J_K$ ) be the ideal generated by  $p$  and  $q$  in  $A$  (respectively,  $B$ ). If  $p$  and  $q$  have an irreducible common divisor in  $B$ , it generates a height 1 prime ideal  $I$ , such that  $J_K \subseteq I$ . Contraction preserves height for integral extensions, so  $I \cap A$  is a height 1 prime ideal in  $A$ . As  $A$  is the polynomial algebra  $k[X]$ , a height 1 prime ideal is a principal ideal generated by an irreducible polynomial. Since  $J_k \subseteq I \cap A$ , that polynomial divides both  $p$  and  $q$  in  $A = k[X]$ , contradicting the hypothesis that they are relatively prime.  $\square$

**Lemma 16.** *Reduced fractions remain reduced for any field extension  $K/k$ .*

*Proof.* If  $K/k$  is algebraic, this is just a restatement of Lemma 15. Suppose now that  $K/k$  is pure transcendental. Factor numerator and denominator into irreducible polynomials in  $k[X]$ . In  $K[X]$  the polynomials are irreducible by Lemma 14 and any potential cancellation has quotient in  $k$  by Lemma 13, thus contradicting the hypothesis that the original fraction is reduced. The general case follows immediately, since any field extension is an algebraic extension of a pure transcendental extension.  $\square$

*Example 2.* The real polynomial fraction  $1/(x^2 + 1)$  is defined over  $\mathbb{Q}$  and reduced over  $\mathbb{C}$ .

**Theorem 3.1.** *(Consistency) Let  $f \in K(X)$  be a rational function in  $n > 0$  variables defined over a field  $K$ . Suppose that  $f$  is defined over a subfield  $k \subset K$ . Then*

- (1)  $f = p/q$ , where  $p$  and  $q$  are relatively prime polynomials in  $K[X]$  that have coefficients in  $k$ .
- (2)  $f$  is defined at  $x \in K^n$  if, and only if,  $q(x) \neq 0$ .
- (3)  $f(x)$  is defined if, and only if, it is defined viewing  $k$  as the coefficient field and  $K$  as a commutative  $k$ -algebra; if so, the value is the same.

- (4) if defined,  $y = f(x) \in K$  belongs to the subfield generated by  $k$  and the coordinates of  $x$ .

*Proof.* To prove the first conclusion, express  $f$  as a reduced fraction  $p/q$  with  $p, q \in k[X]$ , then apply Lemma 16. The second is an earlier noted property of reduced fractions, easily proved by unique factorization. For the third, use  $p/q$  in both cases and observe that the test and conditional value computation are identical. Fourth,  $p(x)$  and  $q(x)$  obviously belong to that subfield.  $\square$

*Remark.* Although more cumbersome to state, the results for  $x \in K^n$  extend to points with coordinates in a commutative  $K$ -algebra.

The most important idea in the above theorem is that, although polynomials can acquire new factorizations when the coefficient field is extended, rational functions cannot acquire new value definitions. That is, a rational function either remains undefined at a point or retains its prior value, no matter how it is expressed or simplified in the new context.

Over  $\mathbb{R}$ , there is an entirely different way, involving analytic functions, to determine if a rational function is defined at a point. Let  $U$  be an open subset of  $\mathbb{R}^n$  and  $f$  a real analytic function defined on  $U$ . If  $x \in \mathbb{R}^n$  is a point on the boundary of  $U$ , then  $f$  is said to *extend analytically* to  $x$  if there exists a real analytic function  $g$ , defined on an open neighborhood  $V$  of  $x$ , such that  $f = g$  on  $U \cap V$ .

**Lemma 17.** *Let  $f \in \mathbb{R}(X)$  be a real rational function and write it as  $f = p/q$ , where  $p$  and  $q$  are real polynomials with no common nonconstant real polynomial divisor. As a real valued function,  $f$  is well defined and real analytic on the open set  $U$  where  $q \neq 0$ . If  $x \in \mathbb{R}^n$  and  $q(x) = 0$ , then  $x$  is a boundary point of  $U$ , but  $f$  cannot be analytically extended to  $x$ .*

*Proof.* All clear, except the issue of extending  $f$  to  $x$ . Let  $g$  be a real analytic extension to  $x$ . Let  $B$  be a small Euclidean ball around  $x$  in  $\mathbb{C}^n$  on which the power series expansion for  $g$  at  $x$  converges absolutely and uniformly, defining a complex analytic function  $\tilde{g}$  on  $B$  that restricts to  $g$  on  $B' = B \cap \mathbb{R}^n$ . Since  $p = gq$  on  $B' \cap U$ , the same relation holds for the power series expansions at  $x$ . So  $p = \tilde{g}q$  on  $B$ . By Lemma 15,  $p$  and  $q$  are relatively prime as complex polynomials. That can also be proved, somewhat more simply, by using complex conjugation. So the complex hypersurfaces  $p = 0$  and  $q = 0$  have no common irreducible components. Since complex hypersurfaces, unlike real hypersurfaces, cannot have isolated points, there exist points  $x' \in B$  arbitrarily close to  $x$  that satisfy  $q(x') = 0$  and  $p(x') \neq 0$ . That contradicts  $p = \tilde{g}q$ . Even if attention is restricted to points at which  $q$  is nonzero, the values of  $\tilde{g} = p/q$  on such points would not be bounded in any neighborhood of  $x$ . That contradicts the analyticity of  $\tilde{g}$ , proving that the presumed analytic function  $g$  cannot exist.  $\square$

*Example 3.* The real rational function  $f = (x^4 + y^4)/(x^2 + y^2)$  is not defined at the origin  $(0,0)$ . At every other point of the  $(x,y)$ -plane it is defined and satisfies  $0 < f(x,y) \leq x^2 + y^2$ . So setting  $f(0,0) = 0$  yields a unique continuous extension of  $f$  to the entire plane. Although that extension is locally bounded at the origin, it is not real analytic there.

### 3.2. The birational case.

In this section a prefix, such as  $k$ -, will signal the specific field (or ring) of coefficients under consideration. The prefix will be omitted if clear from the context or irrelevant, and the terms 'real' and 'complex' will usually be used instead of  $\mathbb{R}$ - and  $\mathbb{C}$ -.

**Proposition 18.** *Let  $K$  be a field and  $F : K^n \rightarrow K^n$  a rational map. Suppose  $F$  is defined over a subfield  $k \subset K$ . Then  $F$  is  $k$ -birational ( $k(F) = k(X)$ ) if, and only if, it is  $K$ -birational ( $K(F) = K(X)$ ).*

*Proof.* The birationality condition  $k(F) = k(X)$  is equivalent to the assertion that  $k(X)$  has dimension 1 as a vector space over  $k(F)$ . But that dimension is invariant under faithfully flat extension, as when tensoring over  $k$  with any field extension  $K$ .  $\square$

So if  $k \subseteq \mathbb{R}$ , then  $k$ -birationality is also equivalent to  $\mathbb{R}(F) = \mathbb{R}(X)$  or  $\mathbb{C}(F) = \mathbb{C}(X)$ . That is, if  $F$  has a purely algebraic rational inverse with complex coefficients (ignoring any issues about where the inverse is defined in  $\mathbb{C}^n$  or in  $\mathbb{R}^n$ ), then it has one with real coefficients that belong to  $k$ .

**Lemma 19.** *Let  $K$  be a field and  $f : K^n \rightarrow K$  an everywhere defined rational function. Suppose  $f$  is defined over a subfield  $k \subset K$ . Then  $f$  restricts to an everywhere defined  $k$ -rational function from  $k^n$  to  $k$ .*

*Proof.* Considering  $f$  as a  $k$ -rational function, the consistency assertions of Theorem 3.1 imply that its domain of definition is  $K^n \cap k^n = k^n$  and that its values there (necessarily in  $k$ ) are those it had as a  $K$ -rational function.  $\square$

A similar conclusion applies to an everywhere defined map with any finite number of rational function components. While a rational bijection  $K^n \rightarrow K^n$  defined over a subfield  $k \subset K$  will restrict to an injection  $k^n \rightarrow k^n$ , that map does not have to be surjective.

*Example 4.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $y = f(x) = x + x^3$ . The restriction of  $f$  to  $\mathbb{Q}$  is a map into, but not onto,  $\mathbb{Q}$ . For instance,  $y = 1$  is not a value by the rational root test.

**Lemma 20.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous open semi-algebraic map. If  $F$  is injective on a Zariski open subset of  $\mathbb{R}^n$ , then it is injective on all of  $\mathbb{R}^n$ .*

*Proof.* Let  $U$  be a Zariski open subset of  $\mathbb{R}^n$ , such that  $F$  is injective on  $U$ . The complement of  $U$  in  $\mathbb{R}^n$  is semi-algebraic (indeed algebraic), of maximum dimension at most  $n - 1$ . By the general form of the Tarski-Seidenberg projection property, its image under  $F$  is also semi-algebraic of maximum dimension at most  $n - 1$ . So it is not Zariski dense. Let  $Z$  be its Zariski closure. The set  $F^{-1}(Z)$  is semi-algebraic. It also has empty interior, as otherwise  $Z$  would contain an open set. So it has maximum dimension at most  $n - 1$ , and hence is not Zariski dense. Let  $Z'$  be its Zariski closure, and let  $U' = U \setminus Z'$ . Then  $U'$  is a nonempty Zariski open subset of  $\mathbb{R}^n$  and for every  $x \in U'$ , the point  $y = F(x)$  has only one inverse image anywhere in  $\mathbb{R}^n$ . If  $F$  is not injective, let  $a, b \in \mathbb{R}^n$  satisfy  $a \neq b$  and  $F(a) = F(b)$ . Take disjoint open sets  $U_a$  and  $U_b$ , such that  $a \in U_a$  and  $b \in U_b$ . Since  $F(U_a) \cap F(U_b)$  is open and  $F$  is continuous, one can shrink  $U_a$  and  $U_b$  so that they also satisfy  $F(U_a) = F(U_b)$ . So for any  $x \in U_a$ , the point  $y = F(x)$  has at least two inverse images. Since  $U'$  is Zariski open,  $U_a \cap U' \neq \emptyset$ , and any point of intersection yields a contradiction.  $\square$



Remark. The final part of the above proof is a specific case of arguments about the counting function  $m(x)$  in section 3.

**Lemma 21.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map. If  $F$  is both birational and an open map, then  $F$  is injective.*

*Proof.* By birationality, there exist Zariski open subsets  $U$  and  $V$  of  $\mathbb{R}^n$ , such that  $F$  maps  $U$  bijectively onto  $V$ . As  $F$  is injective on  $U$ , it satisfies the hypotheses of Lemma 20, so is injective on  $\mathbb{R}^n$ .  $\square$

**Theorem 3.2.** *([Birational case of the RRJC]) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map with nowhere vanishing Jacobian determinant. If  $F$  is birational, then it has an everywhere defined real rational inverse. And in that case, if  $F$  is defined over a subfield  $k \subset \mathbb{R}$ , then its restriction to  $k^n$  is a  $k$ -rational everywhere defined bijection of  $k^n$  onto  $k^n$ , and that also holds for its inverse.*

*Proof.*  $F$  is an open map, so it is injective by Lemma 21. Hence it is surjective by the Białynicki-Birula and Rosenlicht Theorem [13]. Since it is locally bianalytic,  $F$  has a global inverse,  $F^{-1}$ , that is real analytic on all of  $\mathbb{R}^n$ .  $F^{-1}$  is a real analytic extension to  $\mathbb{R}^n$  of the rational inverse of  $F$ . By Lemma 17, each component of  $F^{-1}$  is an everywhere defined real rational function and so  $F^{-1}$  is an everywhere defined real rational map. If  $F$  is defined over a subfield  $k \subset \mathbb{R}$ , start with a rational inverse with coefficients in  $k$ , as is possible by Proposition 18. Argue as before, then apply Lemma 19 componentwise to both  $F$  and  $F^{-1}$ .  $\square$

Remark. In [19], polynomial maps  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that map  $\mathbb{R}^n$  bijectively onto  $\mathbb{R}^n$  are considered, and the question is raised of when the inverse is rational. If so, the inverse is everywhere defined on  $\mathbb{R}^n$  and  $F$  is called a polynomial-rational bijection (PRB) of  $\mathbb{R}^n$ . A key technical result is that a polynomial bijection is a PRB if its natural extension to a polynomial map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  maps only real points to real points. A PRB  $F$  has a nowhere vanishing Jacobian determinant  $j(F)$ . Conversely, it is shown that a nowhere vanishing  $j(F)$  alone suffices to establish that a polynomial map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  of degree two is a bijection and a PRB. A related but stronger condition is defined and shown to be sufficient, but not necessary, for polynomial maps of degree greater than two.

### 3.3. Promoted SRJC cases.

This section is primarily concerned with some known special cases in which the SRJC holds on the basis of topological considerations implying injectivity, and which therefore generalize effortlessly to the RRJC context. The special cases appear in the previously cited paper [11] by Zbigniew Jelonek and in a fairly recent note [20] by Christopher I. Byrnes and Anders Lindquist.

Let  $F : X \rightarrow Y$  be a continuous map of topological manifolds. Let  $A(F)$  be the set of points  $y \in Y$  at which  $F$  is not proper, and let  $B(F) = F^{-1}(A(F))$ . Recall that  $A(F)$  is closed, that the restriction of  $F$  to the induced map  $X \setminus B(F) \rightarrow Y \setminus A(F)$  is proper, and that  $A(F)$  is the smallest subset of  $Y$  with those properties. If  $F(X)$  is open, then its topological boundary  $\partial F(X)$  is contained in  $A(F)$ . In nice enough cases, removing subsets of codimension  $i$  does not affect homotopy groups in dimensions less than  $i - 1$ . Indeed, in [11, Lemma 8.1], Jelonek proves that if  $A$  is a closed semi-algebraic subset of  $\mathbb{R}^n$ , then  $\mathbb{R}^n \setminus A$  is connected if  $A$  is of codimension greater than one and simply connected if it is of codimension greater

than two. The usual conventions apply, namely that the empty set has dimension  $-\infty$  and codimension  $+\infty$ .

**Theorem 3.3.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a real rational everywhere defined map with nowhere vanishing Jacobian determinant. Let  $A(F)$  be the set of points in the codomain at which  $F$  is not proper. Then the following are equivalent:*

- (1)  $F$  has a global real analytic inverse,
- (2)  $A(F) = \emptyset$ ,
- (3)  $\dim(A(F)) < n - 2$ ,
- (4)  $A(F) \cap F(\mathbb{R}^n) = \emptyset$ ,
- (5)  $A(F) = \partial F(\mathbb{R}^n)$ .

*Proof.* The well known so-called topological Hadamard theorem states that a local homeomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  is a homeomorphism if, and only if, it is proper. That yields the equivalence of (1) and (2) in the current context, since the Jacobian condition implies that  $F$  is locally real bianalytic. The equivalence therefore does not require the surjectivity theorem (ST) of [13] for injective rational maps  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . The essential points are that a proper local homeomorphism of connected manifolds (necessarily of the same dimension) is a covering map (necessarily surjective) and that the base,  $\mathbb{R}^n$ , is simply connected and so has no nontrivial connected cover.

Obviously, (2) implies (3) and (4). It implies (5) as well, because (1) implies that  $\partial F(\mathbb{R}^n) = \emptyset$ .

In case (3), let  $B(F) = F^{-1}(A(F))$ . The induced map  $\mathbb{R}^n \setminus B(F) \rightarrow \mathbb{R}^n \setminus A(F)$  is a proper local homeomorphism. Since  $\dim(A(F)) < n - 2$ , the base is simply connected. Because  $F$  is a local homeomorphism,  $\dim(B(F)) < n - 2$ . So the total space is certainly connected. It follows that  $F$  is injective on  $\mathbb{R}^n \setminus B(F)$ .  $B(F)$  is not Zariski dense, so  $F$  is injective on a Zariski open subset of  $\mathbb{R}^n$ . By Lemma 20 it is injective on  $\mathbb{R}^n$ . Finally, use the ST to conclude that  $F$  is also surjective and therefore (1) holds. The result that (3) implies (1) in the SRJC context (that is, for polynomial maps with nowhere vanishing Jacobian determinant) is precisely what is proved in [11, Thm 8.2], and Jelonek's proof is the model for the one presented here.

In case (4), since  $A(F)$  is contained in the closure of the image of  $F$ , the condition  $A(F) \cap F(\mathbb{R}^n) = \emptyset$  amounts to saying that the map of  $\mathbb{R}^n$  onto its image is proper. The main result of [20] is that the standard complex JC holds for polynomial maps that are proper as maps onto their image. In a remark at the end of the note, (4) is proved to imply (1) in the SRJC context. Briefly, (4) implies that  $\mathbb{R}^n$  is a universal covering space, of finite degree  $d$ , of  $F(\mathbb{R}^n)$ . By well known results of the branch of topology called P. A. Smith theory, there are no fixed point free homeomorphisms of  $\mathbb{R}^n$  onto itself of prime period. But the fundamental group  $\pi_1(F(\mathbb{R}^n))$  is of order  $d$ , and contains an element of prime period unless  $d = 1$ . So  $d = 1$ ,  $F$  is injective, and (1) follows as in case (3), by using the ST. The assumption that  $F$  is polynomial, rather than just real analytic, is used at only two points in the proof. First, it ensures that the degree of the covering map is finite. Second, it allows the ST to be applied. Rationality is sufficient in both situations, so (4) implies (1) in the RRJC context as well.

Case (5) is equivalent to case (4) because  $F(\mathbb{R}^n)$  is open in  $\mathbb{R}^n$ , hence  $\partial F(\mathbb{R}^n) \subseteq A(F)$  and  $A(F)$  is contained in the disjoint union of  $F(\mathbb{R}^n)$  and  $\partial F(\mathbb{R}^n)$ .  $\square$

Apart from the two special cases above, there are some closely related issues worth noting. Continue to assume that  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a real rational everywhere defined map with nowhere vanishing Jacobian determinant, and, recall the general properties of such maps described in section 3.  $A(F)$  has positive codimension and  $\mathbb{R}^n \setminus A(F)$  is the disjoint union of finitely many connected open subsets of  $\mathbb{R}^n$ , each of which is either entirely contained in the image of  $F$  or has empty inverse image. Clearly  $F(\mathbb{R}^n)$  is dense in  $\mathbb{R}^n$  if, and only if, none of them has an empty inverse image. If  $\dim(A(F)) < n - 1$  then  $\mathbb{R}^n \setminus A(F)$  is connected, so it has only a single connected component and  $F$  has dense image. The codimension at least two condition (CD2) is of particular interest for dimension  $n = 2$ . In that case  $A(F)$  is either empty or a finite set of points. If  $F$  is polynomial, then  $A(F)$  is  $\mathbb{R}$ -uniruled, only  $A(F) = \emptyset$  is possible, and so the SRJC holds if CD2 and  $n = 2$  are true [11, Section 8]. That line of reasoning is not available for rational maps.

The condition that the image of  $F$  is dense (DI) is strictly weaker than CD2, as shown by considering Pinchuk maps. But even so, it has important implications for the coimage,  $\mathbb{R}^n \setminus F(\mathbb{R}^n)$  of  $F$ . The coimage is always closed and semi-algebraic. If DI is true, then each connected component of, and hence all of, the complement of  $A(F)$  is contained in the image of  $F$ . Equivalently, the coimage of  $F$  is contained in  $A(F)$ . Since  $A(F)$  has codimension at least one, so does the coimage. No example is known for which DI is false, in which case the coimage would have codimension zero. Of course, if the Jacobian condition is dropped, there are examples aplenty, such as  $y = f(x) = x^2$ . Combining several results yields

**Theorem 3.4.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a real rational everywhere defined map with nowhere vanishing Jacobian determinant. Let  $A(F)$  be the set of points in the codomain at which  $F$  is not proper, and let  $B(F) = F^{-1}(A(F))$ . Then  $F$  has dense image if, and only if, the coimage of  $F$  is contained in  $A(F)$ . And in that case,  $A(F)$  is the disjoint union of the coimage and  $F(B(F))$ . If  $F$  has dense image, but is not surjective, then the coimage and  $F(B(F))$  are both nonempty, so the coimage is a nonempty, closed, semi-algebraic, proper subset of  $A(F)$ .*

*Proof.* If  $F$  has dense image, then the preceding paragraph shows that the coimage, which is always closed and semi-algebraic, is contained in  $A(F)$ . The converse is clear. It is nonempty if  $F$  is not surjective. If it were all of  $A(F)$ , then  $A(F)$  would be disjoint from the image of  $F$ , and so  $F$  would have an inverse by Theorem 3.3; and hence be surjective. If  $y \in A(F)$ , then it is either in the image of  $F$ , and so is in  $F(B(F))$ , or it is not, and so is in the coimage.  $\square$

In the complex JC context, it is well known that the coimage has complex codimension at least two. Briefly, the reasoning is as follows. Since the coimage is closed and constructible, if it has codimension less than two it contains an irreducible hypersurface  $h = 0$ ,  $h \circ F$  vanishes nowhere and so is constant, contradicting the algebraic independence of the components of  $F$ . In the SRJC and RRJC contexts, there are no parallel results for the real codimension of the coimage, even if the map has dense image.

### 3.4. Dense images.

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map. There is a simple algebraic criterion that guarantees that  $F$  has a dense image, and it works even with a weaker Jacobian condition. In this section, drop the assumption that

$j(F)$  vanishes nowhere on  $\mathbb{R}^n$ , but do require that  $j(F)$  is not identically zero on  $\mathbb{R}^n$ . As shown in section 3, that is enough to ensure that the components of  $F$  are algebraically independent and hence  $\mathbb{R}(X)$  is an algebraic extension of  $\mathbb{R}(F)$  of finite degree. To facilitate a geometric view of this situation, introduce coordinates  $y_1, \dots, y_n$  in the codomain of  $F$  and think of the map as given by  $y_i = f_i(x_1, \dots, x_n)$ , identifying  $\mathbb{R}(y_1, \dots, y_n) = \mathbb{R}(Y)$  with  $\mathbb{R}(F)$ . Let  $d$  be the degree of the field extension, and  $t \in \mathbb{R}(X)$  a primitive element (generator) over  $\mathbb{R}(Y)$ . Then  $t$  is a root in  $\mathbb{R}(X)$  of a degree  $d$  irreducible polynomial  $R(T)$  with coefficients in the polynomial ring  $\mathbb{R}[Y]$ , such that no nonconstant polynomial in  $\mathbb{R}[Y]$  is a common divisor of all the coefficients.  $R(T)$  is unique up to a nonzero real constant factor, and is obtained from the monic minimal polynomial of  $t$  over  $\mathbb{R}(Y)$  by writing its coefficients as reduced fractions and then multiplying the whole polynomial by a least common multiple of the denominators. Write  $R(y)(T)$  for the polynomial with real coefficients obtained by evaluating the coefficients at a point  $y \in \mathbb{R}^n$ .  $R$  is also irreducible in  $\mathbb{R}[Y, T]$ , and so its set of zeros is an affine irreducible variety  $V$  in  $\mathbb{R}^{n+1}$ . Use  $y_1, \dots, y_n, z$  as coordinates in  $\mathbb{R}^{n+1}$ . Clearly,  $F$  factors as the rational map  $(F, t) : \mathbb{R}^n \rightarrow V$  followed by the projection  $p_n : V \rightarrow \mathbb{R}^n$  onto the first  $n$  components.  $(F, t)$  is actually birational (by construction), but not necessarily everywhere defined, because  $z = t(x)$  may not be defined at all points  $x \in \mathbb{R}^n$ . The projection map  $p_n : V \rightarrow \mathbb{R}^n$  is regular (equivalently, polynomial), as it corresponds to the algebra homomorphism  $\mathbb{R}[Y] \subseteq \mathbb{R}[V] = \mathbb{R}[Y, T]/(R(T))$ . The discriminant  $D$  of  $R(T)$  lies in the coefficient ring  $\mathbb{R}[Y]$ . Up to a nonzero factor in  $\mathbb{R}(Y)$ , it is the product of the squares of the differences of the roots of  $R(T)$  in a splitting field. By Galois theory all the roots are primitive elements. The derivative of  $R(T)$  with respect to  $T$ , a polynomial of degree  $d - 1$  in  $T$ , would be zero at a repeated root. So all the roots are simple, and so  $D$  is nonzero. There is a universal formula for the discriminant of a polynomial in one variable of a given fixed degree in terms of its coefficients. However, the formula applies only if the degree is actual, not just formal; that is, the leading coefficient must be nonzero. So  $D(y)$  is the discriminant of  $R(y)(T)$ , provided that  $y$  is a point at which the leading coefficient of  $R(T)$  does not vanish.

**Lemma 22.** *There is a nonempty Zariski open subset  $U$  of  $\mathbb{R}^n$ , such that for each  $y \in U$  all the following hold:*

- (1)  $R(y)(T)$  has degree  $d$ ,
- (2) the roots, real or complex, of  $R(y)(T)$  are distinct ( $D(y) \neq 0$ ),
- (3) the number of inverse images of  $y$  under  $F$  equals the number of real roots of  $R(y)(T)$ ,
- (4) for each  $x \in \mathbb{R}^n$  with  $y = F(x)$ ,  $t(x)$  is defined, and it is a different real root of  $R(y)(T)$  for each different  $x$ .

*Proof.* The function field of the variety  $V \subset \mathbb{R}^{n+1}$  is  $\mathbb{R}(Y)[t] = \mathbb{R}(F)[t]$ , so  $(F, t)$  is birational. Let  $A$  and  $B$  be Zariski open subsets of  $\mathbb{R}^n$  and  $V$ , respectively, such that  $(F, t)$  is a biregular map of  $A$  onto  $B$ . The image of  $\mathbb{R}^n \setminus A$  under  $F$  and of  $V \setminus B$  under  $p_n$  are both semi-algebraic subsets of  $\mathbb{R}^n$  of maximum dimension at most  $n - 1$ . The Zariski closure of their union is therefore an algebraic set of maximum dimension at most  $n - 1$ . Let  $U$  be its complement. Then  $U$  is nonempty, Zariski open,  $F^{-1}(U) \subseteq A$ , and  $p_n^{-1}(U) \subseteq B$ .  $U$  will be modified in the course of this proof. First, further restrict  $U$  by requiring that both the leading coefficient and discriminant of  $R(T)$  not have any zeros on  $U$ . That takes care of (1) and (2).

If  $y \in U$  and  $z$  is a real root of  $R(y)(T)$ , then  $(y, z)$  is a point of  $V$  that lies in  $B$ . So  $b = (y, z)$  is the image of a point  $x \in A$  under  $(F, t)$ . Since  $(F, t)$  is regular on  $A$ , this implies that  $t$  is defined at  $x$  and  $t(x) = z$ . That shows that the number of inverse images  $x$  is at least as large as the number of real roots  $z$ . As  $F^{-1}(U) \subseteq A$ ,  $t$  is defined at any inverse image  $x \in \mathbb{R}^n$  and  $z = t(x)$  is a surjective map of inverse images to real roots. The final step in the proof is to further restrict  $U$  so as to ensure that the correspondence is bijective. The  $n$  coordinate polynomials  $x_i \in \mathbb{R}(X) = \mathbb{R}(Y)[t]$  are each (uniquely) polynomials of degree less than  $d$  in  $t$  with coefficients in  $\mathbb{R}(Y)$ . Restrict  $U$  to contain only points at which all the coefficients of those polynomials are defined. Then for  $y \in U$ , not only are  $y = F(x)$  and  $z = t(x)$  functions of  $x$ , but also  $x$  is a function of  $y$  and  $z$ .  $\square$

This leads directly to the following theorem. Note that the hypotheses on  $F$  imply not only that the function field extension is algebraic of finite degree, but also that  $F$  is bianalytic at some point, ensuring that the image of  $F$  contains an open set and is thus at least Zariski dense. The theorem is moderately practical in application, allowing one to deal with a single specific polynomial in one variable.

**Theorem 3.5.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map. Assume that the Jacobian determinant of  $F$  is not identically zero on  $\mathbb{R}^n$ . Let  $t$  be a primitive element for the associated function field extension, and  $m$  its monic minimal polynomial. Then the following are equivalent:*

- (1)  $F(\mathbb{R}^n)$  is dense in  $\mathbb{R}^n$ ,
- (2)  $F(\mathbb{R}^n)$  contains a Zariski open subset of  $\mathbb{R}^n$ ,
- (3)  $m$  is defined and has at least one real root on a Zariski open subset of  $\mathbb{R}^n$ ,
- (4)  $m$  has at least one real root everywhere it is defined.

*Proof.* The polynomial  $R(T) \in \mathbb{R}(Y)[T]$  of the preceding lemma, divided by its leading coefficient, is the monic minimal polynomial of  $t$ . The latter is defined exactly at the points  $y \in \mathbb{R}^n$  at which  $R(y)(T)$  has full degree, because there is no common divisor of all the coefficients of  $R(T)$ . And at those points both polynomials have the same roots. Since  $m$  is defined everywhere except at the zeros of the leading coefficient of  $R(T)$ , (4) implies (3). If (3) holds for a Zariski open  $V \subseteq \mathbb{R}^n$ , the preceding lemma implies that the Zariski open subset  $U \cap V$  is contained in the image of  $F$ , proving (2). Obviously, (2) implies (1). The final step is to show that (1) implies (4). Assume (4) is false. Take  $y_0 \in \mathbb{R}^n$  at which  $m$  is defined, but none of the roots are real. As long as the degree remains constant, the roots of a polynomial, here  $R(y_0)(T)$ , depend continuously on its coefficients. So there is an open neighborhood of  $y_0$  on which  $m$  is defined and has only complex roots. That open set intersects  $U$  in a nonempty open set. By the lemma, that intersection lies outside the image of  $F$ , contradicting (1).  $\square$

**Corollary 23.** *For  $F$  as above, if the field extension is of odd degree, then the image of  $F$  is dense.*

*Proof.* A polynomial of odd degree has a real root.  $\square$

*Example 5.* For  $y = f(x) = x^2$  and  $x$  chosen for  $t$ ,  $R(T) = T^2 - y$ , which is irreducible over  $\mathbb{R}(Y)$ , but factors over  $\mathbb{R}(X)$  as  $(T - x)(T + x)$ . Note that its specializations  $R(y)(T)$  for  $y < 0$  do not factor over  $\mathbb{R}$ .

### 3.5. An injectivity criterion.

Start this section with the same notations and assumptions as in the preceding one. In particular,  $j(F)$  may have zeros, but does not vanish identically on  $\mathbb{R}^n$ . However,  $t$  will no longer be a completely arbitrary primitive element, but instead will be selected using the following well known results.

**Lemma 24.** *Some real linear combination  $c_1x_1 + c_2x_2 + \cdots + c_nx_n, c_i \in \mathbb{R}$  of the coordinate polynomials is primitive.*

*Proof.* Choose  $c_1 = 1$ . For some  $c_2 \in \mathbb{R}$ , the subfield of  $\mathbb{R}(X)$  generated over  $\mathbb{R}(F)$  by the single element  $x_1 + c_2x_2$  is the same as the subfield generated by the two elements  $x_1$  and  $x_2$ . For otherwise, since there are only finitely many intermediate fields in characteristic zero, there would be combinations  $x_1 + c_2x_2$  with different values of  $c_2$  that lie in the same proper subfield, which must therefore contain  $x_2$  and hence also  $x_1$ , a contradiction. The ultimate result follows by adding one summand at a time, to obtain a linear combination that generates  $\mathbb{R}(X) = \mathbb{R}(x_1, \dots, x_n)$ , and so is primitive by definition.  $\square$

Remark. Almost all linear combinations are primitive. More specifically, the coefficients  $(c_1, c_2, \dots, c_n) \in \mathbb{R}^n$  of those that are not primitive belong to a finite union of proper vector subspaces of  $\mathbb{R}^n$ , one subspace for each proper subfield of  $\mathbb{R}(X)$  containing  $\mathbb{R}(F)$ .

Recall the identification of  $y_i$  with  $f_i$ , and hence of  $\mathbb{R}[Y]$  and  $\mathbb{R}[F]$ .

**Lemma 25.** *If  $t$  is a primitive element, there is a multiple  $at$  of it, for a polynomial  $a \in \mathbb{R}[Y]$ , such that  $at$  is primitive and its monic minimal polynomial has coefficients in  $\mathbb{R}[Y]$ .*

*Proof.* Let  $R(T) \in \mathbb{R}[Y][T]$  be the minimal degree polynomial with root  $t$  of the previous section, and  $a \in \mathbb{R}[Y]$  the coefficient of its leading term as a polynomial of degree  $d$  in  $T$ . From  $R(t) = 0$ , it follows that  $at^d$  is a sum of terms of lower degree in  $t$ . But then  $(at)^d = a^{d-1}(at^d)$  can be written as a sum of terms of lower degree in  $(at)$ . That yields a degree  $d$  monic polynomial  $S(T) \in \mathbb{R}[Y][T]$ , such that  $S(at) = 0$ . Since  $0 \neq a \in \mathbb{R}(Y)$ , the field generated by  $at$  over  $\mathbb{R}(Y)$  is also  $\mathbb{R}(X)$ . So  $S$  is the monic minimal polynomial of  $at$ .  $\square$

**Theorem 3.6.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map with a Jacobian determinant that is not identically zero on  $\mathbb{R}^n$ . Then there exists a primitive element  $t$  for the associated function field extension  $\mathbb{R}(F) \subseteq \mathbb{R}(X)$ , such that*

- (1)  $t$  is defined everywhere on  $\mathbb{R}^n$ , and
- (2)  $t$  is integral over  $\mathbb{R}[F]$ .

*Proof.* Use the two lemmas in succession. After the first, one has a polynomial primitive element. If  $t$  is a polynomial ( $t \in \mathbb{R}[X]$ ), the same is not necessarily true of  $at$ , since  $a$  is a polynomial in the components of  $F$ , which are not assumed to be polynomial. But  $at$  will be an everywhere defined real rational function. Rename it  $t$ .  $\square$

Assume in the following that  $t$  satisfies (1) and (2).  $R(T)$  will denote some nonzero multiple, by a real constant, of the monic minimal polynomial of  $t$ . Condition (1) ensures that for all  $x \in \mathbb{R}^n$  there is a corresponding real root  $t(x)$  of  $R(y)(T)$

at  $y = F(x)$ . Condition (2) implies that for all  $y \in \mathbb{R}^n$  the roots of  $R(y)(T)$  are continuous functions of  $y$ . One way of stating continuity of roots more precisely is that there exist  $d$  continuous functions  $r_i : \mathbb{R}^n \rightarrow \mathbb{C}$ , such that  $R(y)(r_i(y)) = 0$  for any  $y \in \mathbb{R}^n$ , and all roots are obtained in this way. A canonical way to define the  $r_i$  is to totally order  $\mathbb{C}$  using a lexicographic ordering of the real and complex parts, and then let  $r_i(y)$  be the  $i^{\text{th}}$  element of the set of all  $d$  roots of  $R(y)(T)$ , where the roots are repeated according to multiplicity and arranged in order from smallest to largest [21]. The set of all distinct roots for all  $y \in \mathbb{R}^n$  is then the closed subset of  $\mathbb{R}^n \times \mathbb{C}$  that is the union of the graphs of the continuous functions  $r_i$ ; the individual function graphs will also be called sheets. Each sheet is closed in  $\mathbb{R}^n \times \mathbb{C}$  and its projection to  $\mathbb{R}^n$  is a homeomorphism. The projection of the set of roots onto  $\mathbb{R}^n$  is clearly proper. A consequence is that roots over points sufficiently close to a given point  $y$  are each near a uniquely determined distinct root over  $y$ . It is obvious that repeated roots are those that lie on more than one sheet, and it follows from the above consequence of properness that the multiplicity of a repeated root is exactly the number of sheets on which it lies.

*Example 6.* Let  $R(y)(T) = T^2 - (y^2 + y^4)T + y^6 = (T - y^2)(T - y^4)$ . Since all the roots are real, they are ordered in the usual way if the above method is used, but the slightly unnatural sheets  $r_1 = \min(y^2, y^4)$  and  $r_2 = \max(y^2, y^4)$  are produced, rather than the natural algebraic sheets  $y^2$  and  $y^4$ .

Denote by  $\#r$  ( $\#c$ ) the number of real (respectively, complex) roots counted with multiplicities. Here, a complex root is understood to be a root with a nonzero complex part. For any integer  $i$ , the condition  $\#c \geq i$  (equivalently,  $\#c > i - 1$ ) is open, meaning that the set of points  $y \in \mathbb{R}^n$  at which it holds is an open set, simply because  $\mathbb{R}$  is a closed subset of  $\mathbb{C}$ . And  $\#r \geq i$  (equivalently,  $\#r > i - 1$ ) is closed, as it is the negation of  $\#c > d - i$ . Also, any real root with arbitrarily close complex roots is repeated, since complex conjugate roots lie on different sheets.

One can generalize Theorem 3.5 as follows. Let  $E_1$  be the closure of the image of  $F$ . Given that  $j(F)$  does not vanish identically, the image of  $F$  contains some interior points, and hence so does  $E_1$ . Let  $O$  be the complement of  $E_1$  in  $\mathbb{R}^n$ , and  $E_2$  the closure of  $O$ .  $E_2$  is empty if, and only if,  $F$  has dense image, and otherwise contains some interior points. If  $F$  does not have dense image, then  $E_1$ ,  $E_2$ , and  $E_1 \cap E_2$  are all closed and nonempty, since  $\mathbb{R}^n$  is connected.

**Proposition 26.** *If  $y \in E_1$ , then  $R(y)(T)$  has at least one real root. If  $y \in E_2$ , then every real root of  $R(y)(T)$  has multiplicity greater than one.*

*Proof.* If  $y = F(x)$ , then  $t(x)$  is a real root over  $y$ . Since  $\#r \geq 1$  is a closed condition, it holds on  $E_1$ . By Lemma 22 of section 3.4, there is a Zariski open subset  $U \cap O$  of  $O$  consisting of points over which  $R(T)$  has only complex roots. So every real root over a point of  $E_2$  has arbitrarily close complex roots.  $\square$

The notion of sheets can be used to prove a significant criterion for injectivity. Some prerequisite terminology follows. A map of topological spaces is said to be locally injective at a point if it is injective on some neighborhood of that point. By the celebrated Invariance of Domain Theorem of Brouwer, a continuous injective map of an open subset of  $\mathbb{R}^n$  to  $\mathbb{R}^n$  has an image that is an open subset of  $\mathbb{R}^n$ , and is a homeomorphism onto that image. Since  $F$  is continuous, then if it is locally injective at a point, it is also an open map at that point.

**Lemma 27.** *Suppose  $a \neq b$  are points in  $\mathbb{R}^n$ , and  $F$  is locally injective at both  $a$  and  $b$ . If  $F(a) = F(b) = y$  and  $t(a) = t(b) = r$ , then  $r$  is a repeated root of  $R(y)(T)$ .*

*Proof.* Suppose, to the contrary, that  $r$  is a simple root. Then there is an open subset  $O$  of the set of roots (or of  $\mathbb{R}^n \times \mathbb{C}$ , for that matter) that contains  $(y, r)$  and no points on any other sheet than the sheet on which  $(y, r)$  lies. The inverse image of  $O$  under the map  $(F(x), t(x))$  is an open subset of  $\mathbb{R}^n$  that contains both  $a$  and  $b$ . By the local injectivity hypothesis, it contains two disjoint open sets  $U_a$  and  $U_b$ , containing  $a$  and  $b$ , respectively, such that  $F$  is injective, hence open, on each of  $U_a$  and  $U_b$ .  $F(U_a) \cap F(U_b)$  is therefore an open neighborhood of  $y$ . By Lemma 22, it contains a point  $y'$ , such that real roots over  $y'$  correspond bijectively to inverse images of  $y'$  under  $F$ . Take an inverse image of  $y'$  in  $U_a$  and one, necessarily different, in  $U_b$ . The corresponding roots are then distinct. That contradicts the fact that their images under  $(F(x), t(x))$  both lie in  $O$ , and hence on a single sheet.  $\square$

Remark. Local injectivity is used in the proof only to deduce that  $F$  is open at  $a$  and  $b$ . Still, it seems the appropriate hypothesis in attempts to prove injectivity on a larger scale.

Note that if  $A \subseteq \mathbb{R}^n$ , then, unless  $A$  is open, the assertion that  $F$  is locally injective at every point  $a \in A$  is not a statement about the values of  $F$  on  $A$ , but rather about its values on open neighborhoods of the points of  $A$  in  $\mathbb{R}^n$ . The terminology for roots and sheets will be slightly simplified. At a point  $x \in \mathbb{R}^n$ ,  $t$  will be said to determine a simple (repeated) root if  $(F(x), t(x))$  lies on only one (respectively, more than one) sheet, without any explicit reference to the polynomial  $R(F(x)(T))$  of which  $t(x)$  is a root.

**Theorem 3.7.** (*Injectivity Criterion*) *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map with a Jacobian determinant that is not identically zero on  $\mathbb{R}^n$ . Let  $t$  be an everywhere defined primitive element for the associated function field extension  $\mathbb{R}(F) \subseteq \mathbb{R}(X)$ , such that  $t$  is integral over  $\mathbb{R}[F]$ . Suppose that  $A \subseteq \mathbb{R}^n$  is connected, that  $F$  is locally injective on (some open neighborhood of)  $A$ , and that  $t$  determines only simple roots on  $A$ . Then  $F$  is injective on  $A$ .*

*Proof.* Let  $s$  be the restriction to  $A$  of the continuous map  $(F(x), t(x))$  to the set of roots. The inverse images of the sheets are closed subsets of  $A$ . If two of them intersect at a point  $a$ , then  $s(a)$  lies on more than one sheet, and so  $t(a)$  is a repeated root. But that contradicts the hypotheses, so there can be no such point of intersection. Since  $A$  is connected, only one of the disjoint closed subsets is nonempty. Hence  $s(A)$  is contained in a single sheet. Now suppose that  $a, b \in A$ , with  $a \neq b$ , but  $F(a) = F(b) = y$ . The points  $(y, t(a))$  and  $(y, t(b))$  lie on the same sheet and have the same projection to  $\mathbb{R}^n$ , and so are the same point. That is,  $t(a) = t(b) = r$ . But then  $r$  is a repeated root by Lemma 27, a contradiction. So  $F(a) = F(b)$  is not possible.  $\square$

### 3.6. Applications.

This section concerns some applications of previous results, primarily the injectivity criterion of Theorem 3.7, in the RRJC context. That is, the notations and assumptions of the preceding section apply, but it is also now assumed that  $j(F)$  vanishes nowhere.



Briefly,  $F \in \mathbb{R}(X)^n$  and  $t \in \mathbb{R}(X)$  are defined everywhere,  $j(F)$  vanishes nowhere,  $\mathbb{R}(F)(t) = \mathbb{R}(X)$ , and  $t$  is integral over  $\mathbb{R}[F]$ , with  $R(T) \in \mathbb{R}[Y][T]$  a degree  $d$  nonzero real constant multiple of the monic minimal polynomial of  $t$  over  $\mathbb{R}[Y] = \mathbb{R}[F]$ .

**Lemma 28.** *If  $t$  determines only simple roots on a connected set  $A$ , then  $F$  is injective on  $A$ .*

*Proof.* Follows immediately from the injectivity criterion, since  $F$  is locally bijective everywhere.  $\square$

Let  $R'(T) \in \mathbb{R}[Y][T]$  be the partial derivative of  $R(T)$  with respect to  $T$ . A root  $r$  of  $R(y)(T)$  is a repeated root if, and only if, it is also a root of  $R'(y)(T)$ . Suppose that  $t(x) = r$ . Differentiate the equation  $R(F(x))(t(x)) = 0$  with respect to  $x_1, \dots, x_n$ , obtaining

$$(5) \quad (\nabla_Y R \cdot M + R'(r)v = 0,$$

where  $\nabla_Y R$  is the row  $n$ -vector of partials of  $R(T)$  with respect to the  $Y$  coordinates,  $M$  is the Jacobian matrix of  $F$  at  $x$ ,  $\cdot$  indicates a vector matrix product, and  $v$  is the row  $n$ -vector of partials of  $t$  evaluated at  $x$ . The components of  $\nabla_Y R$  belong to  $\mathbb{R}[Y][T]$ , and in equation 5 they must, of course, be evaluated not only at  $y = F(x)$ , but also at  $T = r = t(x)$ . Write  $\nabla_Y R(x)$  for  $\nabla_Y R(F(x))(t(x))$ . Then  $\nabla_Y R$  is an everywhere defined real rational vector field on the domain of  $F$ .

The case  $d = 1$  is exceptional, since it represents the only situation in which  $t$  can be constant. More on this later. For the moment, just assume that  $t$  is not constant.

**Lemma 29.**  *$\nabla_Y R$  is not the zero vector field, and if  $\nabla_Y R(x) \neq 0$ , then  $t(x)$  is a simple root of  $R(y)(T)$  at  $y = F(x)$ .*

*Proof.* By construction,  $R(T)$  has only simple roots over the Zariski open set  $U$  of Lemma 22.  $F^{-1}(U)$  is a nonempty open set, and it must contain a point at which the gradient vector of  $t(x)$  is nonzero, since  $t(x)$  is a nonconstant real rational function, and so is not locally constant anywhere. Evaluating equation 5 at that  $x$ , the scalar  $R'(r)$  is nonzero, because  $r = t(x)$  is a simple root, and the gradient vector  $v$  is also nonzero. But that is clearly impossible if  $\nabla_Y R(x) = 0$ , and so  $\nabla_Y R$  cannot be the zero vector field. For the second conclusion, since  $M$  is nonsingular at any  $x \in \mathbb{R}^n$ , if  $\nabla_Y R(x) \neq 0$ , then  $R'(r)$  cannot be zero, and hence  $r = t(x)$  is a simple root.  $\square$

**Proposition 30.** *Suppose that  $\nabla_Y R$  has no zeros on a connected set  $A$ . Then  $F$  is injective on  $A$ .*

*Proof.*  $t$  determines only simple roots on  $A$ , by Lemma 29. Now use Lemma 28.  $\square$

Since  $\nabla_Y R$  is an everywhere defined real rational vector field, its set of zeros is a real algebraic set. Points, at which all the components of a vector field vanish, are usually called singular points of the vector field. So the set of zeros of  $\nabla_Y R$  will be denoted by  $S$ . Note that  $S$  is of codimension at least 1 in  $\mathbb{R}^n$ , since  $\nabla_Y R$  is not the zero vector field.

**Corollary 31.** *If  $S$  has codimension 2 or more (in particular, if  $S$  is empty), then  $F$  is globally injective.*

*Proof.* By reason of the codimension assumption,  $A = \mathbb{R}^n \setminus S$  is connected (see section 3.3), so  $F$  is injective on  $A$ . Also,  $A$  is Zariski open, so  $F$  is injective on  $\mathbb{R}^n$  by Lemma 20.  $\square$

Remark. Of course,  $F$  is then invertible. This situation is similar to that in section 3.3, where the codimension of the asymptotic variety was considered. Chances seem better here, since the naive expectation for the dimension of the singular points of a vector field is zero.

Perhaps more practically, one always has injectivity on each connected component of  $\mathbb{R}^n \setminus S$ .

Back to the case  $d = 1$ . This is the birational case, so  $F$  is globally injective. Since  $\mathbb{R}[Y]$  is integrally closed in  $\mathbb{R}(Y) = \mathbb{R}(X)$ ,  $t$  must be a polynomial in  $\mathbb{R}[Y]$ . If that polynomial is a constant, then  $\nabla_Y R$  is identically zero. If not, then  $S$  is the inverse image under  $F$  of the set of zeros of the gradient vector field of  $t \in \mathbb{R}[Y]$  on the codomain.

*Example 7.* This example works out the details for the specific Pinchuk map defined in section 2.1.  $F = (P(x, y), Q(x, y))$  and the auxiliary polynomials  $t, h, f, q, u$  have their meanings here as there. The selected primitive element is  $h$ , and section 2.2 shows that

$$R(h) = (197/4)h^6 + \cdots + (2PQ - 170P^3)h - P^2Q = 0.$$

for a polynomial  $R(T) \in \mathbb{R}[P, Q][T]$ , which is fully written out in the appendix (section 4, equation 6). The companion vector field is  $(\partial R/\partial P, \partial R/\partial Q)$ , evaluated at  $T = h$ .

$R$  has only three terms involving  $Q$  making it easy to compute  $\partial R/\partial Q = -(T - P)^2$ . The expression for  $\partial R/\partial P$  is considerably more complicated, but on substituting  $T$  for  $P$ , it simplifies to  $T^3(6T^2 + 14T + 8)$ , which is conveniently independent of  $Q$ . At any singular point of the vector field, therefore, both  $P = h$  and  $h^3(6h^2 + 14h + 8) = 0$ , must hold. And, conversely, any such point belongs to  $S$ , the set of all singular points of the vector field.

$S$  can be determined more specifically from the equations  $t = xy - 1, h = t(xt + 1), f = (xt + 1)^2(t^2 + y), P = f + h$  that define  $P$ . And  $F(S)$  can be determined by evaluating  $Q = q - u$ , where  $q = -t^2 - 6th(h + 1)$  and  $u = u(f, h)$  is defined by equation 1 in section 2.1. If  $P = h$ , then  $f = (xt + 1)^2(t^2 + y) = 0$ . The case  $xt + 1 = x^2y - x + 1 = 0$  is equivalent to  $x \neq 0$  and  $y = 1/x - 1/x^2$  and it is easy to show that  $P = h = 0$  and  $Q = -1/x^2$  on the two curves. The case  $t^2 + y = x^2y^2 - 2xy + y + 1 = 0$  is equivalent to  $y < 0$  and  $x = 1/y \pm 1/\sqrt{-y}$ , and on these two curves  $P = h = -1$  and  $Q = y - u(0, -1) = y - 163/4$ . Since both 0 and  $-1$  are roots of  $h^3(6h^2 + 14h + 8)$ ,  $S$  is the union of these four curves. They are disjoint, since the curves in each pair are disjoint and the image under  $F$  of each curve in the first case is the negative  $Q$ -axis, while in the second case it is the portion of the line  $P = -1$  satisfying  $-\infty < Q < -163/4$ . All four branches are asymptotic to the  $y$ -axis at  $y = -\infty$  and to the  $x$ -axis at either  $x = -\infty$  (three times), or at  $x = +\infty$  (case 1,  $x > 0$ ). Also,  $y$  is bounded above on all branches, with a maximum value  $y = 1/4$  at  $x = 2$  (case 1).

$S^c = \mathbb{R}^2 \setminus S$  is the disjoint union of five unbounded connected open sets. Each region has a boundary consisting of one or two branches of  $S$ . By Proposition 30,  $F$  is injective on each of those five regions. Recall that  $B(F) = F^{-1}(A(F))$  consists of three curves, and its complement of four regions, with  $F$  injective on each

region. Although the closure of  $F(S)$  intersects  $A(F)$  (at  $(-1, -163/4)$  and  $(0, 0)$ ),  $F(S) \cap A(F) = \emptyset$ . So each of the curves composing  $B(F)$  lies in  $S^c$  and, since it is connected, in just one of its five component regions. In fact, the component curve of  $B(F)$  on which  $-1 < P < 0$  lies in the region of  $S^c$  on which  $y$  is not bounded above (the 'top' region), whereas the other two curves lie in the adjacent region bounded by the two branches of  $S$   $x = 1/y + 1/\sqrt{-y}$  ( $y < 0$ ) and  $y = 1/x - 1/x^2$  ( $x < 0$ ). These assertions can be checked by adding a few more curves with known images to  $B(F)$ , so as to form connected configurations of curves, with images that do not intersect  $F(S)$ , and then determining the location of a single point in each configuration relative to the branches of  $S$ . For details, see the appendix.

Dually, each branch of  $S$  is contained in one of the four regions of  $\mathbb{R}^2 \setminus B(F)$ . Two of the regions each map diffeomorphically onto the connected component of  $\mathbb{R}^2 \setminus A(F)$  containing the positive  $P$ -axis (see the figure in section 2.1). Since their image region contains no point of  $F(S)$ , they cannot contain any branch of  $S$ . Both the case 1,  $x < 0$  and the case 2,  $x = 1/y - 1/\sqrt{-y}$  branches lie in the same region, because there is no component curve of  $B(F)$  to separate them in the region of  $S^c$  that they bound. Finally, the other two branches of  $S$  must lie in the remaining, fourth, region, because they have the same images as the two in the third region and  $F$  is injective on each region.

### 3.7. The Galois case.

As before, let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map with nowhere vanishing Jacobian determinant,  $A(F)$  the set of points at which  $F$  is not proper, and  $B(F) = F^{-1}(A(F))$ . Let  $G$  be the group (under composition) of real birational maps  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , such that the corresponding automorphism  $g^*$  of  $\mathbb{R}(X)$  belongs to the group  $G^*$  of automorphisms of  $\mathbb{R}(X)/\mathbb{R}(F)$ , that is, such that  $g^*$  preserves every element of  $\mathbb{R}(F)$ .  $G$  and  $G^*$  are opposite groups; that is, abstractly the same except for a reversal of the order of the product operation. By construction, every  $g \in G$  satisfies  $F \circ g = F$ .

**Lemma 32.** *Any  $g \in G$  is completely determined by its value at any one point at which it is defined.*

*Proof.* Let  $a \in \mathbb{R}^n$  be a point at which  $g$  is defined, and let  $b = g(a) \in \mathbb{R}^n$  be its value there. Let  $U_a, U_b$  be open sets containing  $a$  and  $b$ , respectively, that are mapped homeomorphically by  $F$ , with  $g$  defined on  $U_a$ . Since  $g$  satisfies  $F(g(x)) = F(x)$  at points where it is defined,  $g$  is completely determined on the open set  $U_a \cap g^{-1}(U_b)$ . The components of  $g$  are rational functions on  $\mathbb{R}^n$ , each determined by its restriction as a rational function to any open subset of  $\mathbb{R}^n$ .  $\square$

Let  $W$  be the complement of  $B(F)$  in the domain of  $F$ . Recall, from section 3, that  $W = \mathbb{R}^n \setminus B(F)$  is an open semi-algebraic subset of  $\mathbb{R}^n$ , that the same is true for each of the finitely many connected components  $V$  of  $W$ , and that, moreover, each such  $V$  is a connected cover of finite degree of its image  $U = F(V)$  via the map induced by  $F$ .

**Lemma 33.** *Each  $g \in G$  is defined at every point of  $W$ , and maps  $W$  homeomorphically onto  $W$ .*

*Proof.* The set of points where  $g$  is defined is Zariski open, hence it intersects each  $V$ . Suppose  $g$  is defined at  $a \in V$ , but not at  $b \in V$ . Let  $c(t)$  be a continuous curve

$[0, 1] \rightarrow V$  wit  $c(0) = a$  and  $c(1) = b$ . Replacing  $b$ , if necessary, by the first point on the curve after  $a$  at which  $g$  is not defined, one can assume that  $b$  is a boundary point of the points of definition of one or more of the components of  $g$ , with  $g$  defined at all the other points on the curve. Lift the curve  $F(c(t))$  in  $U = F(V)$  to a curve starting at  $a' = g(a)$  in whatever total space  $V'$  contains  $a'$ . There is no guarantee either that  $V$  and  $V'$  are the same or that they are different, only that they share the same base space  $U$ . Let  $b' \in V'$  be the endpoint of the lifted curve. Using two open sets, one containing  $b$  and the other  $b'$ , both mapped bianalytically by  $F$  onto the same open subset of  $U$ , and employing a slight variation of the argument in Lemma 32, it is clear that  $g$ , and thus each of its components, can be analytically extended to  $b$ . By Lemma 17 in section 3.1, that is a contradiction. Therefore, there is no point  $b \in V$  at which  $g$  is not defined, and since  $V$  was any connected component of  $W$ , it follows that  $g$  is defined on  $W$ . If  $a \in W$ , then  $g(a) \in W$ , because  $F(g(a)) = F(a)$  does not belong to  $A(F)$ . The same considerations apply to the inverse element of  $g$  in the group  $G$ , so  $g$  is a homeomorphism of  $W$  onto  $W$ .  $\square$

**Proposition 34.**  *$G$  acts freely on  $W$  as a finite transformation group. In particular, no element of  $G$ , except the identity, has a fixed point, and the size of the orbit of any point is the number of elements (the order) of  $G$ .*

*Proof.* Combine the two preceding lemmas.  $\square$

**Corollary 35.** *The map of  $W$  onto  $F(W) = F(\mathbb{R}^n) \setminus A(F)$  is exactly  $d$ -to-1.*

*Proof.* Clear.  $\square$

**Proposition 36.** *If  $F$  has an inverse, then the identity is the only automorphism of  $\mathbb{R}(X)$  that preserves every element of  $\mathbb{R}(F)$ .*

*Proof.*  $F$  maps all the points of an orbit to the same point, so if  $F$  is injective then  $G$  must be trivial.  $\square$

That is a necessary condition on the extension for the existence of an inverse for  $F$ . By Theorem 2.1 in section 2.4, any Pinchuk map shows that it is not sufficient.

Call  $F$  Galois over  $\mathbb{R}$ , or just Galois for short, if the extension  $\mathbb{R}(X)/\mathbb{R}(F)$  is Galois. If  $F$  is defined over a subfield  $k \subset \mathbb{R}$ , similarly define 'Galois over  $k$ ' for  $F$  and note that it implies that  $F$  is Galois over  $\mathbb{R}$ , and that the extensions are of the same degree with canonically isomorphic Galois groups. If  $F$  is Galois, then  $G$  is the opposite group of the Galois group.

**Theorem 3.8.** *(Galois case of the RRJC) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a real rational everywhere defined map with nowhere vanishing Jacobian determinant. Suppose that  $F$  is Galois. Then the following are equivalent:*

- (1)  $F$  has a global real analytic inverse,
- (2)  $F$  has an everywhere defined rational inverse,
- (3)  $F$  is birational,
- (4) the Galois group is trivial.

*Proof.* Use Proposition 36 and Theorem 3.2 (the birational case) to prove implications in the order 1,4,3,2,1.  $\square$

That is not the hoped for result. A full analogue of the known Galois case (polynomial maps with nonzero constant Jacobian determinant) would be that the equivalent conditions in the above theorem must be true. In other words, that any Galois extension in this situation is of degree one.

There are some special results in the Galois case, reported here without proof,  $B(F)$  is algebraic, not just closed semi-algebraic. If  $y$  lies in the closure of the image of  $F$ , then  $y \in A(F)$  if, and only if, it has fewer than  $d$  inverse images. Let  $t$  and  $R(T)$  be as in the two immediately preceding sections. Under the same conditions on  $Y$ , (a) all the roots of  $R(y)(T)$  are real, and (b) if all  $d$  roots are distinct, they are the values of  $t$  at  $d$  distinct inverse images of  $y$ . Note that if  $F$  has dense image, then any  $y \in \mathbb{R}^n$  qualifies, and note that only distinct roots are needed in (b), with no further requirement that  $y$  be generic. Unfortunately, these special properties shed no light on the question of whether the extension must be birational.

*Example 8.* Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the map with components the  $n$  elementary symmetric functions in the variables  $x_1, \dots, x_n$ . The extension is Galois, but the Jacobian condition is not met. This example is useful for geometric visualization of the group action.

### 3.8. Modified conjectures.

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map, with a nowhere vanishing Jacobian determinant. As in section 3, the introductory section on real Jacobian conjectures, let  $d$  be the degree of the associated function field extension  $\mathbb{R}(X)/\mathbb{R}(F)$ , and let  $0 < N \leq d$  be the maximum number of inverse images under  $F$  of any point in the codomain of  $F$ .

**Lemma 37.**  *$d - N$  is even.*

*Proof.* The set of points in the codomain  $\mathbb{R}^n$  with  $N$  inverse images is open (section 3). Let  $t \in \mathbb{R}(X)$  be a primitive element for the extension, and  $m(T) \in \mathbb{R}(f)[T]$  its minimal polynomial over  $\mathbb{R}(F)$ . By Lemma 22 in section 3.4,  $m(T)$  is defined and has distinct roots, with exactly  $N$  of them real, over a nonempty open subset of the codomain  $\mathbb{R}^n$ . It suffices to note that at a point of that subset,  $d - N$  is the number of complex roots, which occur in complex conjugate pairs.  $\square$

**Corollary 38.** *If  $F$  is invertible, then  $d$  is odd.*

*Proof.*  $N = 1$ .  $\square$

That suggests the following

**Conjecture 1.** *((MRRJC) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an everywhere defined real rational map, whose Jacobian determinant vanishes nowhere on  $\mathbb{R}^n$ . If the associated function field extension is of odd degree, then  $F$  has a global real analytic inverse.*

The acronym MRRJC stands for modified rational real Jacobian conjecture. This conjecture is not vacuous, as is shown by the examples  $y = f(x) = x + x^d$  for  $d > 1$  odd. The condition that  $d$  is odd can be replaced by the geometricly more natural condition that  $N$  is odd. These conditions are equivalent and necessary. In the RRJC context, the grandiose conclusion is equivalent to the simple statement that  $F$  is injective, or that  $N = 1$ . Indeed, injectivity implies not only the existence of an inverse, but also that the inverse is both real analytic and semi-algebraic, hence

a Nash diffeomorphism. In sum, the conjecture is that  $N$  odd implies  $N = 1$ . One piece of evidence in favor of the conjecture is that its hypotheses imply, by Theorem 3.5 and Corollary 23, that the image of  $F$  is dense in  $\mathbb{R}^n$  and its complement,  $\mathbb{R}^n \setminus F(\mathbb{R}^n)$ , is contained in a real algebraic strict subset of  $\mathbb{R}^n$ .

Remark. There is another necessary condition for invertibility that applies to the function field extension. Namely, by Proposition 36 in section 3.7, the automorphism group of the extension must be trivial. It has not been included as an additional hypothesis in the MRRJC by deliberate choice, partly because it does not, by itself, even exclude the Pinchuk counterexamples, whose extensions have trivial automorphism groups by Theorem 2.1 in section 2.4.

Turn next to a definition of  $N$ , suitable for this section, in a somewhat more general situation. It is a familiar fact that if either the rationality or the Jacobian condition is dropped, there may be no finite upper bound on the number of inverse images. Traditional examples are  $F = (e^x \cos(y), e^x \sin(y))$  and  $F = (x, xy)$ . For such maps, assign  $N$  the symbolic value  $\infty$ , regardless of specifics of the cardinalities of various fibers. This may occur even if the map is quasifinite, meaning each individual fiber is a finite set. If there is a finite bound, let  $N$  be the least such bound. In this way,  $N$  is defined for any map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  whatsoever, and if it is finite, then it is the (finite) maximum of the (finite) cardinalities of all the fibers of  $F$ .

Two maps,  $F$  and  $G$ , from a topological space  $A$  to another one  $B$ , are called topologically equivalent if  $F = h_B \circ G \circ h_A$ , where  $h_A$  and  $h_B$  are homeomorphisms, respectively of  $A$  to itself and of  $B$  to itself. In other words,  $F$  and  $G$  are the same map up to coordinate changes in the domain and codomain by topological automorphisms. Topological stable equivalence for the set of all maps  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  in all dimensions  $n > 0$  is the equivalence relation generated by (1) topological equivalences, and (2) the equivalence of any map  $F = (f_1, \dots, f_n)$ , and its extension  $(f_1, \dots, f_n, x_{n+1}, \dots, x_m)$  to any larger dimension  $m$ . There are many other types of stable equivalence, such as real analytic or polynomial, each characterized by the type of automorphisms allowed for (global) coordinate changes. Stable equivalence, unqualified, will refer to the least restrictive, purely set theoretic, type, with all bijections allowed as automorphisms. Clearly stable equivalence preserves  $N$ , as defined above. That is, two stably equivalent maps have the same value of  $N$ , whether finite or  $\infty$ . Of course, this stable equivalence does not preserve rationality or even the existence of a Jacobian matrix. Nonetheless, if  $F$  and  $G$  are stably equivalent and both are everywhere defined real rational maps with nowhere vanishing Jacobian determinant, then they are equivalent as far as the conjecture is concerned, since both  $N$  odd and  $N = 1$  are preserved.

For brevity, call  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (1) nondegenerate if  $j(F)$  is not identically zero, (2) nonsingular if  $j(F) \neq 0$  everywhere, and (3) a Keller map if  $j(F)$  is a nonzero constant. These terms are meant to imply that  $J(F)$ , the Jacobian matrix of  $F$ , exists at every point of  $\mathbb{R}^n$ , and can be applied to any such  $F$  if the corresponding restriction on  $j(F)$  is satisfied. For polynomial stable equivalence, the applicable automorphisms are polynomial maps with polynomial inverses, making it obvious that such equivalence preserves (in both directions) each of the above three properties.

There are two classic reductions of the ordinary JC to Yagzhev maps [22, 23] and to Drużkowski maps [24]. A Yagzhev map is a polynomial map of the form  $F =$

$X + H$ , where  $X = (x_1, \dots, x_n)$ , and each component of  $H$  is a cubic homogeneous polynomial in the variables  $x_1, \dots, x_n$ . Yagzhev maps are also called maps of cubic homogeneous type. A Drużkowski map (or map of cubic linear type) is a Yagzhev map, for which the components of  $H$  are cubes of linear forms ( $h_i = l_i^3$ ). In a departure from the convention in some other works, these definitions impose no restriction on  $j(F)$ , beyond the obvious  $j(F)(0) = 1$ . Note, however, that a Yagzhev map  $F = X + H$  is a Keller map if, and only if,  $J(H)$  is nilpotent, since both assertions are just different ways of saying that the formal power series matrix inverse of  $J(F)$  is polynomial.

Reduction theorem proofs use the strategy of transforming an original map into a map of the desired form in a succession of steps that preserve the truth value of certain key properties (and typically increase the number of variables).

For the JC,  $\mathbb{C}$  is usually selected as the ground field and the key properties are the Keller property and the existence of a polynomial inverse. But the strategy and specific steps can be applied more generally than just to polynomial Keller maps and yields, for instance, a reduction of the SRJC to the cubic linear case. [24].

**Historical Note.** At the 1997 conference in Lincoln, Nebraska, to honor the mathematical work of Gary H. Meisters, it was suggested by T. Parthasarathy that the SRJC reduction be attempted for the 1994 counterexample of Pinchuk. The challenge was taken up by Engelbert Hubbers, and in 1999 he demonstrated the existence of a counterexample to the SRJC of cubic linear type, coincidentally in dimension 1999. He started with exactly the specific Pinchuk map of section 2.1, used a computer algebra system to verify a human guided reduction path to a Yagzhev map in dimension 203, then explicitly computed a Gorni-Zampieri pairing [25] to a Drużkowski map in dimension 1999, using sparse matrix representations as necessary. These details are excerpted from a comprehensive unpublished note by Hubbers, which he made available.

If the MRRJC is considered only for polynomial maps, it becomes the MSRJC, a modified strong real Jacobian conjecture. Because of the long standing and continuing interest in the SRJC, a separate full statement is warranted.

**Conjecture 2. (MSRJC)** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a real polynomial map, whose Jacobian determinant vanishes nowhere on  $\mathbb{R}^n$ . Suppose that the associated function field extension is of odd degree, or, equivalently, that the maximum cardinality of the fibers of  $F$  is odd. Then  $F$  has a global real analytic inverse.*

There are reduction theorems for the M SRJC parallel to those just discussed for the JC. To reduce the MSRJC to the cubic homogeneous case, it suffices to take reduction steps that preserve  $N$  and to transform only nonsingular maps, in which case (2) below can be stated more simply as the equality of the maximum cardinality of the fibers of  $F$  and  $G$ . The following theorem basically follows [24], with simplifications suggested by Michiel de Bondt.

**Theorem 3.9.** *There is an algorithm that transforms a nondegenerate, polynomial map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  into a map  $G$  of cubic homogeneous type, so that*

- (1)  *$F$  is nonsingular if, and only if,  $G$  is nonsingular, and*
- (2) *a finite bound on the cardinality of fibers holds for  $F$  if, and only if, the same bound holds for  $G$ .*

*Proof.*  $N = \infty$  may occur for singular maps, but that does not occasion any problems. In each step below a map  $F$  is replaced by a map  $G$ , which becomes the

new  $F$  for the next step. At each step both  $F$  and  $G$  are nondegenerate, and they satisfy both (1) and (2) above, whether singular or not. For all but one step, that is true automatically, because the step is an equivalence or stable equivalence using polynomial automorphisms.

Step 1. Lower the degree. Suppose  $F = (f_1, \dots, f_n)$ .  $F$  is polynomially stably equivalent to  $(f_1 - (y+a)(z+b), f_2, \dots, f_n, y+a, z+b)$ , where  $a, b$  are polynomials that depend only on  $x_1, \dots, x_n$ . Thus, if a term of  $f_1$  has the form  $ab$ , with  $\deg(a) > 1$  and  $\deg(b) > 1$ , it can be removed at the cost of introducing two new variables and some terms of degree less than  $\deg(ab)$ . Repeating this for terms of maximum degree until there are no more maximal degree terms of the specified form in any component, one finally obtains a polynomial map  $G$  (in a generally much higher dimension), all of whose terms are of degree no more than three. This is a standard algorithm [23, 2]. There is flexibility in the choice of term to remove next, and one can opportunistically remove a product  $ab$  that is not a single term, making choices to reach a cubic map more quickly. This step is a polynomial stable equivalence.

Step 2. Normalize.  $F$  is now cubic and (still) nondegenerate. Let  $n$  be the current dimension. Choose  $x_0 \in \mathbb{R}^n$  with  $j(F)(x_0) \neq 0$ . After suitable translations,  $(J(F)(x_0))^{-1}F$  becomes a cubic map  $G$ , such that  $G(0) = 0$  and  $G'(0) = J(G)(0)$  is the identity matrix  $I$ . This step is an affine (in the vector space sense) equivalence.

Step 3. Replicate. Now  $F = X + Q + C$ , where  $Q$  and  $C$  are, respectively, the quadratic and cubic homogeneous components of  $F$ . Let  $t$  be a new variable, and put  $G = (X + tQ + t^2C, t)$ . This is the step at which nondegeneracy, (1), and (2) will be explicitly verified. Let  $x$  be any point of  $\mathbb{R}^n$ . For  $t \neq 0$ ,  $G(x, t) = (t^{-1}F(tx), t)$ , so  $j(G)(x, t) = j(F)(tx)$ , and by continuity of polynomials,  $j(G)(x, 0) = 1$ . That yields nondegeneracy and (1). For (2), identify  $\mathbb{R}^n$  with  $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ , and observe that each hyperplane  $t = a$  is mapped to itself by a map affinely equivalent to  $F$  for  $a \neq 0$ , and by the identity for  $a = 0$ . This step is generally not a stable equivalence.

Step 4. Final step. Now  $F = (X + tQ + t^2C, t)$ , with  $Q$  quadratic homogeneous and  $C$  cubic homogeneous, and both independent of  $t$ . Define two polynomial automorphisms  $A_1, A_2$  in  $X, Y, t$ , where  $Y$  is a sequence of  $n$  additional variables, by  $A_1 = (X - t^2Y, Y, t)$  and  $A_2 = (X, Y + C, t)$ . Then  $G = A_1 \circ (X + tQ + t^2C, Y, t) \circ A_2$  is the map of cubic homogeneous type  $(X - t^2Y + tQ, Y + C, t)$ . This step is a polynomial stable equivalence.  $\square$

Remark. The theorem and proof are valid over  $\mathbb{C}$  as well as over  $\mathbb{R}$ , and, indeed, more generally. There are also a number of preservation results not stated in the theorem. For instance,  $F$  is a Keller map if, and only if,  $G$  is a Keller map. In particular, applying steps 1 through 4 to a Pinchuk map yields a Yagzhev map  $G$ , for which  $j(G)$  is not constant and  $J(G)$  is not unipotent.

On inquiry, both Gianluca Gorni and Michiel de Bondt confirmed that Gorni-Zampieri pairing preserves  $N$ , and sent proofs. Since any Yagzhev map can be paired to a Drużkowski map, and nonsingularity is preserved in both directions [25], there is a further reduction of the MSRJC to the cubic linear case.

More recently, reductions of the ordinary JC to the symmetric case have been considered, primarily over  $\mathbb{R}$  and  $\mathbb{C}$ . Let  $k$  denote a field of characteristic zero. In the JC world a polynomial map  $F : k^n \rightarrow k^n$  is often called symmetric, in a startling abuse of language, if  $J(F)$  is a symmetric matrix. In that case,  $F$  is the gradient map of a polynomial function  $h : k^n \rightarrow k$  and  $J(F)$  is the Hessian matrix of second



order partial derivatives of  $h$ . So in the symmetric case, the JC becomes the Hessian conjecture (HC), namely that gradient maps of polynomials with constant nonzero Hessian determinant have polynomial inverses. In [26], Guowu Meng proves, among many other results, the equivalence of the JC and the HC, using what he refers to as a trick. Meng's trick is the construction featured in the proof below, and works over any  $k$ . In [27], Michiel de Bondt and Arno van den Essen prove a more targeted reduction over  $\mathbb{C}$ , namely to symmetric Keller Yagzhev maps. The reduction process involves the use of  $\sqrt{-1}$ , and if applied to a real Keller map may yield a Yagzhev map that is not real. Interestingly, they later show that all complex symmetric Keller Drużkowski maps have polynomial inverses [28].

The following theorem reduces the entire MRRJC, not just the MSRJC, to the symmetric case.

**Theorem 3.10.** *Any nonsingular  $\mathcal{C}^2$  map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , can be extended to a  $\mathcal{C}^1$  nonsingular map  $G : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ , such that*

- (1) *the Jacobian matrix of  $G$  is symmetric,*
- (2)  *$F$  is an everywhere defined real rational map if, and only if,  $G$  is,*
- (3)  *$F$  is polynomial if, and only if,  $G$  is, and*
- (4) *a finite bound on the cardinality of fibers holds for  $F$  if, and only if, the same bound holds for  $G$ .*

*Proof.* Suppose  $F = (f_1, \dots, f_n)$ , for twice continuously differentiable functions  $f_i$  in the variables  $x_1, \dots, x_n$ . Use coordinates  $v_1, \dots, v_n, x_1, \dots, x_n$  on  $\mathbb{R}^{2n}$ . Define a real valued function on  $\mathbb{R}^{2n}$  by  $h(v, x) = v_1 f_1 + \dots + v_n f_n$ . Let  $G$  be the gradient of  $h$ . Then  $G(v, x) = (F(x), v \cdot F'(x))$ , where  $\cdot$  denotes a vector matrix product and  $F'$  denotes  $J(F)$ , the Jacobian matrix of  $F$ . Viewing  $G$  as a map from  $\mathbb{R}^{2n}$  to  $\mathbb{R}^{2n}$ , it is  $\mathcal{C}^1$  and also clearly satisfies (2) and (3). It satisfies (1) because  $J(G)$  is the Hessian matrix of a  $\mathcal{C}^2$  function. To show that  $G$  is nonsingular, just note that  $J(G)$  has a leading  $n$  by  $n$  block of zeros, flanked on the right by  $F'$  and below, therefore, by the transpose of  $F'$ , and so  $j(G)(v, x) = (-1)^n (j(F)(x))^2$ . Now  $G(v, x) = (w, y)$  if, and only if,  $F(x) = y$  and  $v \cdot F'(x) = w$ . Since  $F'(x)$  is an invertible matrix at any point  $x$ , there is a bijection between the inverse images of a point  $(w, y) \in \mathbb{R}^{2n}$  under  $G$  and the inverse images of  $y$  under  $F$ , and that clearly establishes (4).  $\square$

Remark. Application to the JC involves noting that  $F$  is Keller if, and only if,  $G$  is, and that  $F$  has a polynomial inverse if, and only if,  $G$  has.

#### 4. APPENDIX - SUPPLEMENTAL DATA

This final section supplies additional data about the map that is at the heart of this article, namely the Pinchuk map  $F$  of total degree 25 defined and described in section 2.1. It includes a discussion of how the geometric behavior was determined, equations for the asymptotic variety as a polynomial curve, and complete details of the minimal polynomial of section 2.2.

The key to the geometry is the following table. The table shows that the number of connected components of a level set  $P = c$  can vary from 2 to 5. That number and the range of  $Q$  on each connected component can be found by parametrizing the locus of zeros of  $P - c$  or of any factors, expressing  $q$  as a function of the parameter, and taking limits. A bit tricky, but important, is the fact that  $q^+ < q^-$  for every line of the table in which they appear. Since  $j(F)$  vanishes nowhere,  $Q$

is monotone on each connected component of a level set. The description of the number of inverse images of various points given in section 2.1 is then easily verified.

$P = c$	Ranges of $Q$ on the components
$c > 0$	$(-\infty, q+), (q+, q-), (q-, +\infty), (-\infty, +\infty)$
$c = 0$	$(0, 208), (-\infty, 0), (0, +\infty), (-\infty, 0), (208, +\infty)$
$-1 < c < 0$	$(-\infty, q+), (q+, +\infty), (-\infty, q-), (q-, +\infty)$
$c = -1$	$(-\infty, -163/4), (-\infty, -163/4), (-163/4, +\infty), (-163/4, +\infty)$
$c < -1$	$(-\infty, +\infty), (-\infty, +\infty)$
Legend: $(a, b)$ denotes the open interval from $a$ to $b$ , with $a < b$ ; $q+ (q-) =$ the value of $Q$ at $h = -1 + \sqrt{1+c}$ (resp., $-1 - \sqrt{1+c}$ );	

TABLE 1. Ranges of  $Q$  on level sets  $P = c$  for Pinchuk's map

Remark. An equivalent table appeared in [3], but a number of incorrect conclusions about  $F$  were drawn from it. The rational parametrization for level sets  $P = c$ , with  $c \neq 0$  and  $c \neq -1$ , also appeared, unfortunately with a typographical error. However, the author used the correct parametrization in deriving the table. There were corrections in the unpublished manuscript [4] and in [5]. The following parametrizations were also used in [3], but not given explicitly there. For  $P = -1$  the parametrizations are  $x = -t^{-1} - t^{-2}, y = -t^2$  for  $t \neq 0$  and  $x = -s^2, y = -s^{-2} + s^{-3} - s^{-4}$  for  $s \neq 0$ . For  $P = 0$  they are  $x = -t^{-1}, y = -t - t^2$  for  $t \neq 0$  (two components) and  $x = -(h+1)h^{-1}(h+2)^{-2}, y = -h(h+1)(h+2)^2$  for  $h \notin \{0, -2\}$  (three components).

The following details about the equations defining  $A(F)$  are reproduced from [6].  $A(F)$  has the bijective polynomial parametrization by  $s \in \mathbb{R}$ :

$$P(s) = s^2 - 1$$

$$Q(s) = -75s^5 + \frac{345}{4}s^4 - 29s^3 + \frac{117}{2}s^2 - \frac{163}{4}$$

and that its points satisfy the minimal polynomial equation

$$(Q - (345/4)P^2 - 231P - 104)^2 = (P + 1)^3(75P + 104)^2.$$

These equations allow the easy computation of the earlier mentioned points  $a$  and  $b$  at which the line  $P = 3$  intersects  $A(F)$  (take  $s = \pm 2$ ), and of the point  $(-104/75, -18928/375)$  (approximately  $(-1.38, -50.47)$ ) in the Zariski closure of  $A(F)$  that does not lie on the curve  $A(F)$  itself. Note that for  $P = c \geq -1$ ,  $s = h + 1$ , where  $h = -1 \pm \sqrt{1+c}$ . This  $s$  has nothing to do with the  $s$  used above to parametrize two components of  $P = -1$ .

In section 2.2 a polynomial  $R(T)$  was defined, but not fully written out. It has degree 6 in  $T$ , coefficients in  $\mathbb{Q}[P, Q]$ , and  $h$  as a root in  $\mathbb{R}[x, y]$ . It was shown in Proposition 5 that  $R(T) = (197/4)m(T)$ , where  $m(T)$  is the monic minimal polynomial of  $h$  over both  $\mathbb{R}[P, Q]$  and  $\mathbb{R}(P, Q)$ . Straightforward computations show that

$$(6) \quad R(T) = (197/4)T^6 + (104 - (363/2)P)T^5 + (63 - 421P + (825/4)P^2)T^4$$

$$+ (-306P + 510P^2 - 75P^3)T^3 + (-Q + 412P^2 - 195P^3)T^2$$

$$+ (2PQ - 170P^3)T - P^2Q,$$

and this formula makes it trivial to evaluate the effect of setting  $P$  and/or  $Q$  equal to zero.

The following partial derivatives of  $R(T)$  were used, but not fully written out, in the example in section 3.6.

$$\begin{aligned}\partial R/\partial P &= (-363/2)T^5 + (-421 + (825/2)P)T^4 \\ &\quad + (-306 + 1020P - 225P^2)T^3 + (824P - 585P^2)T^2 \\ &\quad + (2Q - 510P^2)T - 2PQ\end{aligned}$$

$$\partial R/\partial Q = -T^2 + (2P)T - P^2 = -(T - P)^2$$

If one sets  $P = T$  in the expression for  $\partial R/\partial P$  above, then the terms of degrees 0 and 1 in  $T$  drop out, and the result is  $T^3$  times the following quadratic polynomial in  $T$  alone.

$$\begin{aligned}&T^2(-363/2 + 825/2 - 225) \\ &\quad + T(-421 + 1020 - 585) \\ &\quad + (-306 + 824 - 510) \\ &= 6T^2 + 14T + 8\end{aligned}$$

The same example postponed to this appendix the verification of the location of the three component curves of  $B(F)$  relative to the branches of  $S$ .

The image of the  $y$ -axis is easily shown to be the straight line  $4Q = 200P + 33$ . The line contains the points  $(0, 33/4)$  and  $(-1, -167/4)$  of the  $(P, Q)$ -plane, which lie, respectively, above and below the points  $(0, 0)$  and  $(-1, -163/4)$  of  $A(F)$ . So the line crosses  $A(F)$  between those points. Since the crossing point has only one inverse image, the  $y$ -axis intersects the component curve of  $B(F)$  on which  $P$  is bounded above. Since  $P = y$  on the  $y$ -axis, the intersection point lies in the 'top' region.

The rational parametrization of the level set  $P = 3$ , described in some detail near the beginning of section 2.2, is given, in part, by the curve  $(x(h), y(h))$  for  $h > 3$ . The image of that curve is the entire line  $P = 3$ , which crosses  $A(F)$  twice. So the curve itself crosses both the other components of  $B(F)$ , by the same reasoning as before. At  $h = 4$ , the parametrization yields the point  $A = (-5/441, 441(-17))$  of the  $(x, y)$ -plane. The vertical line  $x = -5/441$  intersects three of the four branches of  $S$ , each at a single point, in the order, from top to bottom, of  $x = 1/y + 1/\sqrt{-y}$  ( $y < 0$ ) first, then  $y = 1/x - 1/x^2$  ( $x < 0$ ), and finally  $x = 1/y - 1/\sqrt{-y}$  ( $y < 0$ ). The point  $A$  is between the first two points of intersection, and the two component curves of  $B(F)$  at issue must lie in the same region of  $S^c$  as  $A$ . Interestingly, the just added curve and the  $y$ -axis do not intersect, even though their images obviously do.

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